QUESTIONS FOR UNIT 2: REPRESENTABLE BISETS AND COHOMOLOGY

1. Routine questions

1. Consider the $(\mathcal{A}_2, \mathcal{A}_2)$-biset specified by the size matrix $\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ Is it a. representable on the left? b. representable on the right?

2. More challenging questions

3. Questions to which I do not know the answer

4. Research questions

16. Consider the category of linear functors $\mathbb{B}^{1,1} \to R \;\text{mod}\;$ where R is a field of characteristic 0, i.e. biset functors defined on bisets that are representable on both sides. Is this category semisimple? More specifically, if the whole category isn't semisimple, what are the full subcategories of $\mathbb{B}^{1,1}$ for which the biset functors are semisimple? Semisimplicity is known to hold on: groups, discrete categories, $\mathcal{A}_2$.

17. Is it true for Hochschild (co)homology of $k\mathcal{C}$, where k is a field and $\mathcal{C}$ is a finite category, that $HH^n(k\mathcal{C}) \cong \mathrm{Hom}(HH_n(k\mathcal{C}), k)$? Background: this duality formula is true for the cohomology of spaces. For general rings HH^n is quite different from HH_n. However, category algebras are not general rings. When $\mathcal{C}$ is a poset, for example, the theorem of Gerstenhaber and Schack says that Hochschild cohomology coincides with ordinary cohomology, so the duality statement is true.

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QUESTIONS FOR UNIT 3: SIMPLE BISET FUNCTORS

1. Routine questions

4. Order categories so that $\mathbf{1} = [1] < [2] < \cdots$ starts the order. Show that $\mathrm{Ess}([n]) = R$ when $n = 1$ and is zero if $n > 1$.

5. Order categories so that $\mathbf{1} < \mathcal{A}_2$ starts the order. Show that $\mathrm{Ess}_{\mathbb{B}^{(1,1)}}(\mathcal{A}_2) = R$.

2. More challenging questions

13. Show that for each of the simple biset functors S defined on $\mathbb{B}^{(1,1)}$, that are non-zero on $\mathcal{A}_2$ and or the discrete categories $[n]$, if $\mathcal{C}$ is a category of minimal size on which $S(\mathcal{C}) \neq 0$, then $S(\mathcal{C})$ is a simple $R\,\mathrm{Out}(\mathcal{C})$-module.

14. The field of biset functors defined on all finite categories seems so vast that it may be a good idea to to focus on categories of particular types, just as we already have a rewarding theory by focusing the definition on groups, or on p-groups. What categories would you choose to specialize on first, and why? Here is a list of possibilities:

- posets; or, more specifically
- Boolean lattices: each is the poset of subsets of a set
- free categories of quivers with no oriented cycles
- the posets $\mathcal{A}_n$
- monoids
- EI-categories (what are they?)
- other ideas?

15. If you set out to investigate simple biset functors defined on some preferred class of categories, which would you study first, and why?

- biset functors defined on $\mathbb{B}^{\mathrm{all,all}} = \mathbb{B}$
- biset functors defined on $\mathbb{B}^{\mathrm{all},1}$ or on $\mathbb{B}^{1,\mathrm{all}}$
- biset functors defined on $\mathbb{B}^{1,1}$

6. Assume that $\mathrm{End}_{\mathbb{B}^{(1,1)}}(\mathcal{A}_2) = \mathrm{Mat}_{2,2}(R) \oplus R$ and that when k is a field then $G_0(k\mathcal{A}_2)$ has dimension 2. Can you show that G_0 has $S_{\mathbf{1},R}$ as a composition factor as biset functors on $\mathbb{B}^{(1,1)}$? Can you show from your knowledge of representations of $\mathcal{A}_2$ that the kernel of $G_0(k\mathcal{A}_2, \oplus) \to$

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$G_0(k\mathcal{A}_2)$ has dimension 1, and so kernel of $G_0(-,\oplus) \to G_0(-)$ has the simple $S_{\mathcal{A}_2,R}$ as a composition factor?

3. Questions to which I do not know the answer

10. Is it true that the kernel of $G_0(-,\oplus) \to G_0(-)$ equals $S_{\mathcal{A}_2,R}$ as biset functors on $\mathbb{B}^{(1,1)}$?

4. Research questions

16. If S is a simple biset functors, and $\mathcal{C}$ is a category of smallest size with $S(\mathcal{C}) \neq 0$, is $S(\mathcal{C})$ a simple module for $\mathrm{Out}(\mathcal{C})$. If so, then identify the simple modules for $\mathrm{Out}(\mathcal{C})$ that parametrize simple functors in this way.

17. Are there simple biset functors S and categories $\mathcal{C}$ so that $\dim S(\mathcal{C}) = \infty$?

18. Are there categories $\mathcal{C}$ with infinitely many simple biset functors S for which $S(\mathcal{C}) \neq 0$?

19. Each simple functor defined on groups extends uniquely to a simple functor defined on all finite categories. We have some partial idea how $S_{\mathbf{1},k}$ extends at this point. What about other simple functors such as the torsion free part of the Dade group (on p-groups) $S_{C_p\times C_p,\mathbb{Q}}$, or any $S_{H,V}$ where $H \neq \mathbf{1}$ is a group and V is a simple $R\,\mathrm{Out}\,H$-module. Is there a poset $\mathcal{P}$ with $S_{H,V}(\mathcal{P}) \neq 0$?

20. On groups, the simple biset functors on $\mathbb{B}^{1,1}$, $\mathbb{B}^{1,\mathrm{all}}$ and $\mathbb{B}^{\mathrm{all},\mathrm{all}}$ are all parametrized the same way (by pairs (H,V) where H is a group and V is a simple $R\,\mathrm{Out}\,H$-module). Is it also true for biset functors on categories, that simple functors on these three categories are all parametrized the same way?

21. What are the composition factors of $G_0(k\mathcal{C},\oplus)$ as a biset functor on $\mathbb{B}$?