1. Routine questions

1. Verify that G-Set is a symmetric monoidal category $\mathbb{S}$ with product $\diamond$, such that finite colimits exist and commute with $- \diamond X$ for all objects X in $\mathbb{S}$.

2. Is it obvious that indecomposable sets for $\mathcal{A}_n = x_1 < x_2 < \cdots < x_n$ biject with rooted trees? Is it obvious that indecomposable sets for the Kronecker quiver with two objects x, y and two morphisms $x \to y$ biject with directed graphs?

3.True or False?: the identity element of $B(\mathcal{C})$ is a representable set if and only if $\mathcal{C}$ has an initial object.

4. Is it obvious that

$$B(\mathcal{C}) = \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \cong \mathbb{Z}\mathbb{N}^{\times}_{\geq 0}$$

is not a finitely generated ring? How many idempotents does it have? Is it obvious that it is not semisimple? Is B_{FI} semisimple?

5. Do you remember the axioms for a symmetric monoidal category?

6. Let $[n]$ be the discrete category with n objects and only the identity morphisms. Calculate $B([n])$. Is it finite or infinite dimensional. Is it semisimple?

7. Show that if the category $\mathcal{C}$ is a group then self-equivalences of $\mathcal{C}$ are the same thing as automorphisms of the group, and that two such self-equivalences are naturally isomorphic if and only if the automorphisms differ by an inner automorphism.

2. More challenging questions

8. Is it true that FI = finite sets with monomorphisms has finite colimits, commuting with direct product? What about finite sets with epimorphisms?

9. Let SS be a category satisfying conditions in the notes, that might replace the category Set of finite sets. Show that there is a functor Set $\to \mathbb{S}$ taking a set with n elements to the coproduct of n copies of $\mathbf{1}$. What needs to be shown to verify this. Show that this means that linear functors on the category of bi-objects in $\mathbb{S}$ can be regarded as biset functors.

10. Find categories $\mathcal{U}, \mathcal{V}$ so that not all $(\mathcal{U}, \mathcal{V})$-bisets can be factored as a product of basic bisets ${}_{\mathcal{C}^F}\mathcal{E}_{G\mathcal{D}}$ with $\mathcal{C}, \mathcal{D}, \mathcal{E}$ all smaller than the largest of $\mathcal{U}, \mathcal{V}$.

11. Do you know what the idempotent completion of a category is? Do you know different names for it. Have you any idea how to prove: If the idempotent completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent then $\mathcal{C}$ and $\mathcal{D}$

are isomorphic in the biset category $\mathbb{B}_R$? (There is a proof in 7.9.4 of volume I of Borceux's book, but you might want to guess a proof.)

12. Find the size matrices of all the indecomposable bisets in $\mathrm{End}_{\mathbb{B}}(\mathcal{A}_2, \mathcal{A}_2)$ such that all the morphisms in $\mathcal{A}_2 \times \mathcal{A}_2^{\mathrm{op}}$ act by monomorphisms. (Do you see what this means?) How many such bisets are there. Compute a multiplication table for these bisets, showing that the product of any two of these bisets is another one. More difficult: show that (when R is a field) this endomorphism ring is semisimple by finding central idempotents, and then a basis for the ring such that the multiplication is the same as that of matrices with only one non-zero entry, which is 1.

3. Questions to which I do not know the answer

13. Is it true that if $\mathcal{C}$ and $\mathcal{D}$ are isomorphic in the biset category then their idempotent completions are equivalent? This is true in the distributor 2-category, and there is a proof in the book by Borceux.

4. Research questions

14. What does $\mathrm{End}_{\mathbb{B}}(\mathcal{C})$ look like for various categories $\mathcal{C}$ that we might be interested in? For example, $\mathcal{C}$ might be one of the posets $\mathcal{A}_n$ (a chain of length n), or different quivers of type $\mathcal{A}_n$ with other orientations, or the poset of subsets of a set of size n, or some other poset. Do these rings satisfy the Krull-Schmidt theory over a field. What are their units? These units contain $\mathrm{Out}\,\mathcal{C}$, but what else? In other words, what do the invertible linear combinations of bisets look like? Is the span of non-invertible bisets an ideal in $\mathrm{End}_{\mathbb{B}}(\mathcal{C})$, with quotient $R\,\mathrm{Out}\,\mathcal{C}$?

15. Do biset functors, perhaps restricted to a favorable clase of categories, admit filtrations that in characteristic zero give them the structure of a highest weight category, or something of this kind?