Biset functors for categories

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Outline:

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Unit 5: Monoidal structures; fibered biset functors

Monoidal structure on the biset category $\mathbb{B}$

Internal tensor product: the direct product of categories $\mathcal{C} \times \mathcal{D}$
Unit: the category $\mathbf{1}$

Contravariant equivalence: $\tau : \mathbb{B} \to \mathbb{B}$ given by $\tau(\mathcal{C}) = \mathcal{C}^{\mathrm{op}}$.

A $(\mathcal{C}, \mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ is the same thing as a $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$-set, also a $\mathcal{D}^{\mathrm{op}} \times (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$-set. This gives ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ the structure of a $(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$-biset ${}_{\mathcal{D}^{\mathrm{op}}}\Omega_{\mathcal{C}^{\mathrm{op}}}$.

Dual objects

For each category $\mathcal{K}$ the $(\mathcal{K},\mathcal{K})$-biset ${}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}$ is a $\mathcal{K}\times\mathcal{K}^{\mathrm{op}}$-set also a counit $(\mathbf{1},\mathcal{K}^{\mathrm{op}}\times\mathcal{K})$-biset ${}_{\mathbf{1}}\mathcal{K}_{\mathcal{K}^{\mathrm{op}}\times\mathcal{K}}$ and a unit $(\mathcal{K}\times\mathcal{K}^{\mathrm{op}},\mathbf{1})$-biset ${}_{\mathcal{K}\times\mathcal{K}^{\mathrm{op}}}\mathcal{K}_{\mathbf{1}}$.

From these we construct a $(\mathcal{K},\mathcal{K}\times\mathcal{K}^{\mathrm{op}}\times\mathcal{K})$-biset

$$ {}_{\mathcal{K}}\Omega_{\mathcal{K}\times\mathcal{K}^{\mathrm{op}}\times\mathcal{K}} = {}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}} \times {}_{\mathbf{1}}\mathcal{K}_{\mathcal{K}^{\mathrm{op}}\times\mathcal{K}} $$

and also a $(\mathcal{K}\times\mathcal{K}^{\mathrm{op}}\times\mathcal{K},\mathcal{K})$-biset

$$ {}_{\mathcal{K}\times\mathcal{K}^{\mathrm{op}}\times\mathcal{K}}\Psi_{\mathcal{K}} = {}_{\mathcal{K}\times\mathcal{K}^{\mathrm{op}}}\mathcal{K}_{\mathbf{1}} \times {}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}, $$

so that if r,s,x,z are objects of $\mathcal{K}$ and y is an object of $\mathcal{K}^{\mathrm{op}}$ we have

$$ \Omega(r,(x,y,z)) = \{(\alpha,\beta) \mid \alpha : x \to r,\ \beta : z \to y\} $$

and

$$ \Psi((x,y,z),s) = \{(\gamma,\delta) \mid \gamma : y \to x,\ \delta : s \to z\}. $$

Lemma

With these definitions

$$\Omega \circ \Psi \cong {}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}.$$

Exchanging the roles of $\mathcal{K}$ *and* $\mathcal{K}^{\mathrm{op}}$ *we obtain similar bisets* Ω' *and* Ψ' *with* $\Omega' \circ \Psi' \cong {}_{\mathcal{K}^{\mathrm{op}}}\mathcal{K}^{\mathrm{op}}_{\mathcal{K}^{\mathrm{op}}}$.

Proof.

The definition of the biset product at a pair of objects (r, s) is

$$(\Omega \circ \Psi)(r, s) = \bigsqcup_{(x,y,z) \in \mathcal{K} \times \mathcal{K}^{\mathrm{op}} \times \mathcal{K}} \Omega(r, (x, y, z)) \times \Psi((x, y, z), s) / \sim$$

We define a morphism of bisets $\Omega \circ \Psi(r, s) \to {}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}(r, s)$ by $((\alpha, \beta), (\gamma, \delta)) \mapsto \alpha\gamma\beta\delta$. This specification is constant on equivalence classes because if (ζ, θ, ψ) is a triple of morphisms in $\mathcal{K} \times \mathcal{K}^{\mathrm{op}} \times \mathcal{K}$, with suitable domains and codomains, then the pairs

$$((\alpha, \beta)(\zeta, \theta, \psi), (\gamma, \delta)) = ((\alpha\zeta, \theta\beta\psi), (\gamma, \delta))$$

and

$$((\alpha, \beta), (\zeta, \theta, \psi)(\gamma, \delta)) = ((\alpha, \beta), (\zeta\gamma\theta, \psi\delta))$$

both map to the same morphism in ${}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}$, namely $\alpha\zeta\gamma\theta\beta\psi\delta$. This means we have defined a morphism of bisets. In the opposite direction we have an inverse map $\Omega \circ \Psi \leftarrow {}_{\mathcal{K}}\mathcal{K}_{\mathcal{K}}$ given at (r, s) by $\nu \mapsto ((1_r, 1_r), (1_r, \nu))$. $\square$

Corollary

1. *Each category $\mathcal{K}$ has a dual object $\mathcal{K}^{\mathrm{op}}$, in the sense of monoidal categories.*
2. *For any category $\mathcal{K}$ and biset functor M, the value $M(\mathcal{K})$ is a direct summand of $M(\mathcal{K} \times \mathcal{K}^{\mathrm{op}} \times \mathcal{K})$.*

Yoneda-Dress construction

If M is a biset functor and $\mathcal{C}$ a category then

$$\mathsf{P}_{\mathcal{K}}(M)(\mathcal{C}) = M(\mathcal{C} \times \mathcal{K}),$$
$$\mathsf{P}_{\mathcal{K}}^{\mathrm{op}}(M)(\mathcal{C}) = M(\mathcal{C} \times \mathcal{K}^{\mathrm{op}}).$$

These are the *Yoneda-Dress construction* of M at $\mathcal{K}$ and the *opposite Yoneda-Dress construction* of M at $\mathcal{K}$.

Definition

For each category $\mathcal{D}$ let $F_{\mathcal{D}}$ denote the *representable biset functor* at $\mathcal{D}$, so $F_{\mathcal{D}}(\mathcal{C}) = \mathrm{Hom}_{\mathbb{B}}(\mathcal{D}, \mathcal{C}) = A(\mathcal{C}, \mathcal{D})$ is the R-linear span of the indecomposable $(\mathcal{C}, \mathcal{D})$-bisets. In particular $F_{\mathbf{1}} = B$ is the Burnside ring functor.

Proposition

We have isomorphisms

$$\mathsf{P}_{\mathcal{K}} F_{\mathcal{D}} \cong F_{\mathcal{K}^{\mathrm{op}} \times \mathcal{D}} \quad \textit{and} \quad \mathsf{P}_{\mathcal{K}}^{\mathrm{op}} F_{\mathcal{D}} \cong F_{\mathcal{K} \times \mathcal{D}}$$

as biset functors. As a particular case, $F_{\mathcal{D}} \cong \mathsf{P}_{\mathcal{D}}^{\mathrm{op}} B$.

Proof.

For any category $\mathcal{C}$ we have

$$(\mathsf{P}_{\mathcal{K}} F_{\mathcal{D}})(\mathcal{C}) \cong F_{\mathcal{D}}(\mathcal{C} \times \mathcal{K}) \cong \mathrm{Hom}_{\mathbb{B}}(\mathcal{D}, \mathcal{C} \times \mathcal{K}),$$

using Yoneda's lemma. This is the Grothendieck group of $\mathcal{C} \times \mathcal{K} \times \mathcal{D}^{\mathrm{op}}$-sets, which is also a description of $F_{\mathcal{K}^{\mathrm{op}} \times \mathcal{D}}(\mathcal{C})$. The remaining statements follow. □

Proposition

For each category $\mathcal{K}$, the functor $P_{\mathcal{K}}^{\mathrm{op}} : \mathcal{F} \to \mathcal{F}$ is both left and right adjoint to $P_{\mathcal{K}}$.

Definition

Let M and N be biset functors. We define the internal Hom of M and N to be

$$\mathcal{H}(M,N) := \mathrm{Hom}_{\mathcal{F}}(M, \mathsf{P}_{\bullet}(N)),$$

using the functoriality of the Yoneda-Dress construction in its subscript to make $\mathcal{H}$ a biset functor.

Proposition

We have $\mathcal{H}(M,N) \cong \mathrm{Hom}_{\mathcal{F}}(\mathsf{P}_\bullet^{\mathrm{op}}(M), N)$ *as biset functors.*

Proof.

Prove it first when $M = F_{\mathcal{D}}$. In this case

$$\begin{aligned}\mathcal{H}(M,N)(\mathcal{C}) &= \mathrm{Hom}_{\mathcal{F}}(F_{\mathcal{D}}, \mathsf{P}_{\mathcal{C}}(N)) \\ &\cong (\mathsf{P}_{\mathcal{C}}(N))(\mathcal{D}) \\ &\cong N(\mathcal{D} \times \mathcal{C}).\end{aligned}$$

using Yoneda's Lemma.
It follows that the result is true whenever M is any projective biset functor, because it is a summand of a direct sum of representable functors.
Deduce the result for arbitrary M using a projective presentation of M and left exactness of $\mathcal{H}$. □

Definition

We define the *internal tensor product* of (left) biset functors L and M to be the biset functor

$$L \otimes M := (\mathrm{P}_{\bullet}(L) \circ \tau) \otimes_{\mathcal{F}} M$$

where the functoriality on $\mathbb{B}$ is obtained via the subscript $\bullet$. Thus the value of $L \otimes M$ at a category $\mathcal{C}$ is

$$(L \otimes M)(\mathcal{C}) = (\mathrm{P}_{\mathcal{C}}(L) \circ \tau) \otimes_{\mathcal{F}} M = L(\tau(-) \times \mathcal{C}) \otimes_{\mathcal{F}} M(-).$$

Theorem

For all biset functors L, M and N we have

1. $\mathrm{Hom}_{\mathcal{F}}(L \otimes M, N) \cong \mathrm{Hom}_{\mathcal{F}}(M, \mathcal{H}(L, N))$, *and*
2. $\mathcal{H}(L \otimes M, N) \cong \mathcal{H}(M, \mathcal{H}(L, N))$.

Corollary

With the product operation $\otimes$ and unit B the category of biset functors $\mathcal{F}$ is a symmetric monoidal tensor category. Furthermore, if P and Q are projective biset functors then $P \otimes Q$ and $\mathcal{H}(P, Q)$ are also projective.