Biset functors for categories

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Outline:

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Unit 1: $\mathcal{C}$-Sets, the Burnside ring, biset functors

Unit 2: Representable bisets and (co)homology as a biset functor

Unit 3: Simple biset functors

Unit 4: Correspondences

Unit 5: Monoidal structures; fibered biset functors

Unit 6: More about the Burnside ring

Correspondences

Given sets X and Y, a correspondence between X and Y is a subset $U \subset X \times Y$.

Given $U \subseteq X \times Y$ and $V \subseteq Y \times Z$ we define

$$UV := \{(x, z) \in X \times Z \mid \exists\, y \in Y \text{ so that } (x, y) \in U \text{ and } (y, z) \in V\}.$$

This is associative. There is an identity.

The correspondence category Rel:
objects are finite sets.
$\mathrm{Hom}_{\mathrm{Rel}}(Y, X) =$ correspondences between X and Y.

A correspondence functor over k is a functor $\mathrm{Rel} \to k$-mod.

Realizing correspondences as functors

Let $U \subseteq X \times Y$ be a correspondence.
For each subset B of Y put

$$^{+}U(B) = \{x \in X \mid \exists\,(x, y) \in R,\ y \in B\}.$$

Let 2^X = the poset of subsets of the set X.

Proposition

$^{+}U : 2^Y \to 2^X$ is an order preserving map (a functor).

We have a functor $\mathrm{Rel} \to \mathrm{Cat}$ that sends $X \mapsto 2^X$ and $U \mapsto {}^{+}U$.

Correspondences determine birepresentable bisets

Theorem

The functor $H : \mathrm{Rel} \to \mathbb{B}$ that sends a set X to the poset 2^X and a correspondence $U \subset Y \times X$ to the biset ${}_{2^Y}2^Y_{(+U)}{}_{2^X}$ embeds Rel in the category $\mathbb{B}^{1,1}$ of birepresentable bisets.

Corollary

Every biset functor on $\mathbb{B}^{1,1}$ restricts to a correspondence functor.

Example

$X = Y = [2] := \{1, 2\}.$ $$2^X = 2^Y = \begin{array}{ccc} \emptyset & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \{2\} & \longrightarrow & \{1,2\} \end{array}$$

$U := \{(1,1), (1,2), (2,2)\} \subset Y \times X$ is a morphism $X \to Y$ in Rel.

${}^{+}U : 2^X \to 2^Y$ has the effect $\begin{bmatrix} \emptyset \\ \{1\} \\ \{2\} \\ \{1,2\} \end{bmatrix} \to \begin{bmatrix} \emptyset \\ \{1\} \\ \{1,2\} \\ \{1,2\} \end{bmatrix}$

We obtain a biset with size matrix: ${}_{2^Y}2^Y_{({}^{+}U)2^X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$

Notice: it is birepresentable: there are 1s down the left column and along the bottom row.

The essential algebras

Put sets in order $[0] < [1] < [2] < \cdots$ and categories in order $2^{[0]} < 2^{[1]} < 2^{[2]} < \cdots$

Recall $I_{\mathcal{C}}^{<}$ = the k-span of the $(\mathcal{C}, \mathcal{C})$-bisets that factor through categories $\mathcal{D}$ with $\mathcal{D} < \mathcal{C}$.
$\mathrm{Ess}(\mathcal{C}) := \mathrm{End}_{\mathbb{B}^{1,1}}(\mathcal{C})/I_{\mathcal{C}}^{<}$ is the essential algebra of $\mathcal{C}$.

There is a similarly defined essential algebra for the linearized correspondence category $R\mathrm{Rel}$.

Consequences

Let $H : \mathrm{Rel} \to \mathbb{B}^{1,1}$ be the functor constructed.

Corollary

The functor H induces a ring homomorphism $\mathrm{Ess}(X) \to \mathrm{Ess}(2^X)$.

Theorem

In fact, this map is an isomorphism.

Corollary

Simple correspondence functors and simple biset functors on the full subcategory of $\mathbb{B}_{1,1}$ *with posets* 2^X *as objects are parametrized in the same way.*