Biset functors for categories

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Outline:

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Recall: simple biset functors on groups

Biset functors form an abelian category, so we have the concept of a simple functor S: the only subfunctors are S and 0.
Simple functors are naturally defined over a field.
A subfunctor T of S means $T(\mathcal{C}) \subset S(\mathcal{C})$ for all categories $\mathcal{C}$.

The simple biset functors on groups are parametrized $S_{H,V}$ where

- a group H, taken up to isomorphism
- a simple representation V of $\mathrm{Out}\, H$.

For every group G, $S_{H,V}(G)$ is either simple or zero as an $\mathrm{End}_{\mathbb{B}}(G)$-module.
H is the unique group G of minimal size on which $S_{H,V}(G) \neq 0$ and $V = S_{H,V}(H)$.

Simple functors and full subcategories of $\mathbb{B}$

Proposition

Let S be a simple biset functor defined on $\mathbb{B}$.
Let $\mathbb{B}'$ be a full subcategory of $\mathbb{B}$ and let T be a simple functor defined on $\mathbb{B}'$.

- *The restriction of S to $\mathbb{B}'$ is either zero or a simple functor.*
- *T extends uniquely to a simple functor on $\mathbb{B}$, whose restriction to $\mathbb{B}'$ is T.*

Corollary

1. *We have a bijection $\{$simple functors on $\mathbb{B}\} \leftrightarrow$ $\{$simple functors on $\mathbb{B}'\} \sqcup \{$simple functors that vanish on $\mathbb{B}'\}$*
2. *For each object $\mathcal{C}$ of $\mathbb{B}$, $S(\mathcal{C})$ is a simple $\mathrm{End}_{\mathbb{B}}(\mathcal{C})$-module.*
3. *For every simple $\mathrm{End}_{\mathbb{B}}$-module U, there is a unique simple S on $\mathbb{B}$ with $S(\mathcal{C}) = U$*

Corollary

If $\mathrm{End}_{\mathbb{B}}(\mathcal{C})$ has n isomorphism types of simple modules, there are n isomorphism types of simple biset functors S with $S(\mathcal{C}) \neq 0$.

Example

$\mathrm{End}_{\mathbb{B}}(\mathbf{1}) = R$ so there is only one simple biset functor non-zero on $\mathbf{1}$, namely $S_{\mathbf{1},R}$

Example

$\mathrm{End}_{\mathbb{B}^{1,1}}(\mathcal{A}_2) \cong \mathrm{Mat}_{2,2} \oplus R$ so there are two simple biset functors for $\mathbb{B}^{1,1}$ that are non-zero on $\mathcal{A}_2$. One of them is $S_{\mathbf{1},k}$. Which one?

Example

The discrete category $[n]$ has $\mathrm{End}_{\mathbb{B}}([n]) \cong \mathrm{Mat}_{n,n}(R)$ so there is one simple biset functor non-zero on $[n]$, the same answer for $\mathbb{B}^{(1,1)}$. Show that it is $S_{\mathbf{1},R}$.

The essential algebra

Choose any well-order on the ($\mathbb{B}$-isomorphism classes of) finite categories.
Let $I_{\mathcal{C}}^{<}$ = the R-span of the $(\mathcal{C}, \mathcal{C})$-bisets that factor through categories $\mathcal{D}$ with $\mathcal{D} < \mathcal{C}$.
Define $\mathrm{Ess}_R(\mathcal{C}) := \mathrm{End}_{\mathbb{B}_R}(\mathcal{C})/I_{\mathcal{C}}^{<}$, the essential algebra of $\mathcal{C}$.

Example

When G is a finite group we have $\mathrm{Ess}(G) \cong R\,\mathrm{Out}\,G$.

Example

Order categories so that $\mathbf{1} = [1] < [2] < \cdots$ starts the order. Then $\mathrm{Ess}([n]) = R$ when $n = 1$ and is zero if $n > 1$. This shows $\mathrm{Ess}(\mathcal{C}) \neq R\,\mathrm{Out}\,\mathcal{C}$ in general.

Example

Order categories so that $\mathbf{1} < \mathcal{A}_2$ starts the order. Then $\mathrm{Ess}_{\mathbb{B}(1,1)}(\mathcal{A}_2) = R$.

A parametrization of simple biset functors

Proposition

Let S be a biset functor and $\mathcal{C}$ a minimal category with $S(\mathcal{C}) \neq 0$. Then $I_{\mathcal{C}}^{<}$ acts as 0 on $S(\mathcal{C})$. The structure of $S(\mathcal{C})$ as an $\mathrm{End}_{\mathbb{B}}(\mathcal{C})$-module is the same as its structure as an $\mathrm{Ess}_R(\mathcal{C})$-module.

Corollary

Simple biset functors biject with pairs $(\mathcal{C}, V)$ where V is a simple $\mathrm{Ess}(\mathcal{C})$-module.