Biset functors for categories

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Outline:

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Unit 1: $\mathcal{C}$-Sets, the Burnside ring, biset functors

Unit 2: Representable bisets and (co)homology as a biset functor

Unit 3: Simple biset functors

Unit 4: Correspondences

Unit 5: Monoidal structures; fibered biset functors

Unit 6: More about the Burnside ring

The category algebra

See P.J. Webb, *An introduction to the representations and cohomology of categories*, pp. 149-173 in: M. Geck, D. Testerman and J. Thévenaz (eds.), Group Representation Theory, EPFL Press (Lausanne) 2007.

Let $R\mathcal{C}$ be the category algebra of $\mathcal{C}$ over R. It is the algebra that is a free R-module with the morphisms of $\mathcal{C}$ as a basis.
If $\mathcal{C}$ is a group, $R\mathcal{C}$ is the group ring.
If $\mathcal{C}$ is a poset, $R\mathcal{C}$ is the (opposite of) the incidence algebra of $\mathcal{C}$.

Theorem (B. Mitchell)

Representations of $\mathcal{C}$ over R may be identified with $R\mathcal{C}$-modules.

Homology and cohomology of a category

These have both a topological and an algebraic definition.

Write $\underline{R}$ for the constant functor $\mathcal{C} \to R\mathrm{-mod}$ with value R.
Write $|\mathcal{C}|$ for the nerve of $\mathcal{C}$. When $\mathcal{C}$ is a poset, this is the order complex of $\mathcal{C}$.
When $\mathcal{C}$ is a group, it is a model for BG.

The cohomology of $\mathcal{C}$ over R is

$$\begin{aligned} H^*(\mathcal{C}, R) &:= \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \\ &\cong H^*(|\mathcal{C}|; R) \end{aligned}$$

The homology of $\mathcal{C}$ over R is

$$\begin{aligned} H_*(\mathcal{C}, R) &:= \mathrm{Tor}_*^{R\mathcal{C}}(\underline{R}, \underline{R}) \\ &\cong H_*(|\mathcal{C}|; R) \end{aligned}$$

Examples of category cohomology

When $\mathcal{C}$ is a group, it is the group cohomology.

When $\mathcal{C}$ is a poset, it is the cohomology of the order complex.

When Δ is a simplicial complex, write $\operatorname{sd}\Delta$ for the poset of simplices of Δ ordered by inclusion. Then the topological (co)homology of Δ is the (co)homology of $\operatorname{sd}\Delta$.

How can group (co)homology be made a biset functor?

Group (co)homology comes with
restriction: $H^*(G, R) \to H^*(H, R)$ and
corestriction: $H_*(H, R) \to H_*(G, R)$ when $H \leq G$.
There is also conjugation and inflation.

Problem: group cohomology $H^*(G, R)$ has no operation of deflation; homology $H_*(G, R)$ has no inflation. If they were biset functors as defined they would have these operations.

Solution: to make group cohomology into a biset functor we use only bisets ${}_G\Omega_H$ that are free on restriction to H. To make group homology a biset functor we only use bisets ${}_G\Omega_H$ that are free on restriction to G.

Next problem: How can we define (co)homology as a biset functor in a uniform, functorial way?

Hochshild homology

We let $(R\mathcal{C})^e := R\mathcal{C} \otimes_R (R\mathcal{C})^{\mathrm{op}}$ be the enveloping algebra of $R\mathcal{C}$. Its modules are the same as $(R\mathcal{C}, R\mathcal{C})$-bimodules.

Hochschild homology is $HH_n(R\mathcal{C}) := \mathrm{Tor}_n^{(R\mathcal{C})^e}(R\mathcal{C}, R\mathcal{C})$.
Hochschild cohomology is $HH^n(R\mathcal{C}) := \mathrm{Ext}^n_{(R\mathcal{C})^e}(R\mathcal{C}, R\mathcal{C})$.

These definitions work because $R\mathcal{C}$ is free as an R-module.

Theorem (Gerstenhaber,Schack)

If $\mathcal{C}$ is a poset then Hochschild (co)homology and category (co)homology of $\mathcal{C}$ coincide.

Representable bisets

A $\mathcal{C}$-set Ω is representable if $\Omega \cong \coprod \mathrm{Hom}(x_i, -)$, for certain objects $x_i \in \mathcal{C}$.

Theorem

If ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ are bisets that are representable on restriction to each side, then so is ${}_{\mathcal{C}}\Omega \circ \Psi_{\mathcal{E}}$.

Let $\mathbb{B}^{1,1}$ be the subcategory of $\mathbb{B}$ obtained by using only bisets that are representable on each side.
$\mathbb{B}^{1,\text{all}}$ uses only bisets representable on the left.
$\mathbb{B}^{\text{all},1}$ uses only bisets representable on the right.

Hochschild Cohomology of a category as a biset functor

Theorem (Bouc, Keller)

Hochschild homology $\mathcal{C} \mapsto HH_*(R\mathcal{C})$ *is a functor on* $\mathbb{B}^{1,all}$.

S. Bouc, *Bimodules, trace généralisée, et transferts en homologie de Hochschild*, preprint (1997).

B. Keller, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), 223–273.

Homology as a biset functor

Theorem (Bouc, Keller)

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Theorem

Let R be a field. Then $\mathcal{C} \mapsto H^*(\mathcal{C}, R)$ *has the structure of a functor on* $\mathbb{B}^{all,1}$ *and* $\mathcal{C} \mapsto H_*(\mathcal{C}, R)$ *has the structure of a functor on* $\mathbb{B}^{1,all}$.

Overview of the proof

Theorem

Let R be a field. Then $\mathcal{C} \mapsto H^*(\mathcal{C}, R)$ has the structure of a functor on $\mathbb{B}^{all,1}$ and $\mathcal{C} \mapsto H_*(\mathcal{C}, R)$ has the structure of a functor on $\mathbb{B}^{1,all}$.

Xu showed that there are canonical maps $H^*(\mathcal{C}, R) \to HH^*(R\mathcal{C}) \to H^*(\mathcal{C}, R)$ with composite the identity, providing a canonical decomposition $HH^*(R\mathcal{C}) = H^*(\mathcal{C}, R) \oplus Y$ for some summand Y. We show that the same is true for homology. For groups this splitting is well known. For categories, the argument goes via the factorization category $F(\mathcal{C})$ introduced by Quillen, and which is homotopy equivalent to $\mathcal{C}$ via a canonical functor.

F. Xu, *Hochschild and ordinary cohomology rings of small categories,* Adv. Math. **219** (2008), 1872-1893.

Overview of the proof

- There are canonical maps $H_*(\mathcal{C}, R) \to HH_*(R\mathcal{C}) \to H_*(\mathcal{C}, R)$ with composite 1.
- We combine this with the construction of Bouc and Keller in Hochschild homology, getting a definition of homology $H_*(\mathcal{C}, R)$ as a biset functor.
- $H_*(\mathcal{C}, R)$ and $H^*(\mathcal{C}, R)$ are the homology and cohomology of a space (the nerve), so over a field one is the dual of the other. This provides a dependence of cohomology as a biset functor.

Overview of the proof

If ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ is representable on the left then the bimodule $R\Omega$ is projective on the left, and we get a map on Hochschild homology $HH_*(R\Omega) : HH_*(R\mathcal{D}) \to HH_*(R\mathcal{C})$ which is functorial in Ω. We combine this with the splitting of Hochschild cohomology for categories due to Xu. He exploits canonical functors between the factorization category $F(\mathcal{C})$ and Hochschild category $\mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$, namely $\mathcal{C} \leftarrow F(\mathcal{C}) \to \mathcal{C}^e$. The first of these is a homotopy equivalence (shown by Quillen) so induces an isomorphism on (co)homology. The second has the property that $\mathrm{Tor}_*^{R\mathcal{C}^e}(R\mathcal{C}, R\mathcal{C}) \cong \mathrm{Tor}_*^{RF(\mathcal{C})}(\underline{R}, R\mathcal{C}\downarrow_{F(\mathcal{C})}))$ and there is a canonical splitting of representations of $F(\mathcal{C})$ which is $\underline{R} \to R\mathcal{C}\downarrow_{F(\mathcal{C})} \to \underline{R}$, so that ordinary homology $H_*(\mathcal{C}, R)$ is canonically a direct summand of Hochschild homology $HH^*(R\mathcal{C})$. We use this splitting to construct the map on $H_*(\mathcal{C}, R)$. For cohomology, we use the fact that $H_*(\mathcal{C}, R)$ is the homology of a space, and so its cohomology over a field is the vector space dual of the homology, giving contravariant dependence on Ω.

Postlude

- In finding a corestriction operation for the cohomology of categories we have shifted the focus away from comparing a category with a subcategory.
- Biset functors on $\mathbb{B}^{1,1}$ are easier to work with than general biset functors. A formula was given in Webb (1993) for the simple such functors and an application given to computing group cohomology. Does this formula extend?
- The construction of the operation on homology coming from a biset is quite complicated. Is there a more direct way, especially on cohomology? Note that the construction avoids use a pair of functors that are adjoint on both sides (unlike, for instance, the transfer of Linckelmann on the Hochschild cohomology of symmetric algebras).