Biset functors for categories

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Outline:

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Unit 1: $\mathcal{C}$-Sets, the Burnside ring, biset functors

Unit 2: Representable bisets and (co)homology as a biset functor

Unit 3: Simple biset functors

Unit 4: Correspondences

Unit 5: Monoidal structures; fibered biset functors

Unit 6: More about the Burnside ring

Sets with an action of a category

$\mathcal{C}$ is a finite category and Set denotes the category of finite sets.
A $\mathcal{C}$-set is a functor $\Omega : \mathcal{C} \to \mathrm{Set}$.

The Burnside ring of $\mathcal{C}$ is
$B(\mathcal{C}) =$ the Grothendieck group finite $\mathcal{C}$-sets with relations
$\Theta = \Omega + \Psi$ if $\Theta \cong \Omega \sqcup \Psi$ as $\mathcal{C}$-sets.
The product of $\mathcal{C}$-sets is defined pointwise:
$(\Omega \cdot \Psi)(x) := \Omega(x) \times \Psi(x)$.

The poset $\mathcal{A}_2 = x < y$

Example

$\mathcal{C} = \mathcal{A}_2 = \underset{x}{\bullet} \xrightarrow{\alpha} \underset{y}{\bullet}$ is the poset $x < y$. The $\sqcup$-indecomposable $\mathcal{C}$-sets have the form

$$\Omega_n := \{1, \ldots, n\} \to \{*\}, \quad n \geq 0.$$

A finite category may have infinitely many non-isomorphic 'transitive' sets.

$$B(\mathcal{C}) = \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \cong \mathbb{Z}\mathbb{N}^{\times}_{\geq 0}$$

The ring is more complicated than we might expect. It is not even finitely generated. It is not semisimple over any ring.

More general actions of a category

More generally, instead of Set, we may consider a symmetric monoidal category $\mathbb{S}$ with product $\diamond$, such that finite colimits exist and commute with $- \diamond X$ for all objects X in $\mathbb{S}$ and that decompositions with respect to $\sqcup$ are unique up to isomorphism.

For example $\mathbb{S}$ could be G-Set, or RG-mod when R is a field, or the category FI of finite sets with monomorphisms, or span categories of these.

A $\mathcal{C}$-object in $\mathbb{S}$ is a functor $\Omega : \mathcal{C} \to \mathbb{S}$. We define $B_{\mathbb{S}}(\mathcal{C})$ by analogy with $B(\mathcal{C})$.
For example, $B_{\mathrm{FI}}(\mathcal{C})$ is a subring of $B(\mathcal{C})$.

Most, and possibly all constructions we shall consider work for $\mathcal{C}$-objects in $\mathbb{S}$.

The Burnside ring of a discrete category

Let $[n]$ be the category with n objects and only identity morphisms.

Exercise: $B([n])$
How big is it? Is it semisimple if we take coefficients to be a field?

Bisets for categories

These are called distributors or profunctors in the literature and were introduced in
J. Bénabou, *Les distributeurs*, 1973.
and appear also in
Marta Bunge, *Categories of Set-Valued Functors*, University of Pennsylvania, 1966.
There is a good account in
F. Borceux, *Handbook of Categorical Algebra I*, Cambridge Univ. Press 1994.

Given categories $\mathcal{C}$ and $\mathcal{D}$ a $\mathcal{C}$-set is a functor $F : \mathcal{C} \to$ finite sets.
A $(\mathcal{C}, \mathcal{D})$-biset is a functor $\Omega : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to$ finite sets.

Composition of bisets

Given a $(\mathcal{C},\mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and a $(\mathcal{D},\mathcal{E})$-biset ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ there is a $(\mathcal{C},\mathcal{E})$-biset $\Omega \circ \Psi$ given by

$$\Omega \circ \Psi(x,z) = \bigsqcup_{y \in \mathcal{D}} \Omega(x,y) \times \Psi(y,z) / \sim$$

where $\sim$ is the equivalence relation generated by $(u\beta, v) \sim (u, \beta v)$ whenever $u \in \Omega(x,y_1)$, $v \in \Psi(y_2,z)$ and $\beta : y_2 \to y_1$ in $\mathcal{D}$.

The biset category with categories as objects

Proposition (Bénabou)

The operation $\circ$ is associative up to isomorphism of bisets. For each category $\mathcal{C}$ there is an identity biset ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$, specified on each pair of objects of $\mathcal{C}$ as ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$.

Let $A_R(\mathcal{C}, \mathcal{D})$ be the free R-module with basis the isomorphism classes of finite $(\mathcal{C}, \mathcal{D})$-bisets that are indecomposable with respect to $\sqcup$.
The biset category $\mathbb{B}$ over R has as objects all finite categories, with homomorphisms $\mathrm{Hom}_{\mathbb{B}}(\mathcal{C}, \mathcal{D}) = A_R(\mathcal{D}, \mathcal{C})$.
A biset functor is an R-linear functor $\mathbb{B} \to R$-mod.
This extends the usual notion of biset functors defined on groups.

A calculation in $\mathbb{B}$

Let $n \in \mathbb{N}$,
$[n]$ = category with object set $\underline{n} = \{1, \ldots, \}$, and only identity morphisms.
An $([m], [n])$-biset is a set for the category $[m] \times [n]^{\mathrm{op}}$ = discrete category with the mn objects (i, j).
A set for it is a list of mn sets S_{ij}. They form an array

$$\Omega = (S_{ij}) = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1m} \\ \vdots & & & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nm} \end{pmatrix}.$$

It is a disjoint union of copies of the bisets E_{ij} which have a one-point set in position (i, j) and are empty elsewhere.
Thus the Grothendieck group $A_{\mathbb{Z}}([m], [n]) \cong \mathrm{Mat}_{m,n}(\mathbb{Z})$.
Composition of bisets is matrix multiplication.
The full subcategory of the biset category $\mathbb{B}_R$ with objects $[m] \simeq$ the category of free modules over R with R-module homomorphisms as morphisms.

The size matrix of a $(\mathcal{C}, \mathcal{D})$-biset Ω

This is the matrix $|\Omega|$ with rows indexed by the objects of $\mathcal{C}$, columns indexed by the objects of $\mathcal{D}$, and where the (x, y) entry is the size $|\Omega(x, y)|$.

Example

Let $\mathcal{C} = \mathcal{E} = \mathcal{A}_2 = 1 \to 2$ and $\mathcal{D} = \mathcal{A}_3 = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ with the objects placed in the order indicated. Consider a $(\mathcal{C}, \mathcal{D})$-biset Ω and a $(\mathcal{D}, \mathcal{E})$-biset Ψ with the size matrices indicated:

$$|\Omega| = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad |\Psi| = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad |\Omega \circ \Psi| = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Note $|\Omega||\Psi| = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$.

Examples of biset functors

Example

For each finite category $\mathcal{C}$ the representable functor $\mathrm{Hom}_{\mathbb{B}}(\mathcal{C}, -)$ is a biset functor. When $\mathcal{C} = 1$ this is the Burnside ring functor B, because $\mathrm{Hom}_{\mathbb{B}}(1, \mathcal{D})$ is the Grothendieck group of $(\mathcal{D}, 1)$-bisets with respect to $\sqcup$, and these bisets are really the same as $\mathcal{D}$-sets.

Example

For each finite dimensional algebra Λ over a field k we let $K_0(\Lambda, \oplus)$ be the Grothendieck group of finite dimensional Λ-modules with respect to direct sum decompositions. For each finite category $\mathcal{C}$ the assignment of $K_0(k\mathcal{C}, \oplus)$ has the structure of a biset functor.

Basic bisets

Definition

Given categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, and functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ we obtain a $(\mathcal{C}, \mathcal{D})$-biset that we denote ${}_{\mathcal{C}^F}\mathcal{E}_{G_{\mathcal{D}}}$. On objects x of $\mathcal{C}$ and y of $\mathcal{D}$ this biset is defined by

$$ {}_{\mathcal{C}^F}\mathcal{E}_{G_{\mathcal{D}}}(x, y) := \mathrm{Hom}_{\mathcal{E}}(G(y), F(x)). $$

The functorial action of $\mathcal{C}$ and $\mathcal{D}$ is given by applying F and G, and composition.
The $(\mathcal{C}, \mathcal{C})$-biset ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ acts as the identity at $\mathcal{C}$.

The Yoneda embedding

Proposition

1. *There is a functor* $\phi : \mathrm{Cat} \to \mathbb{B}_R$ *defined to be the identity on objects, and defined on functors* $F : \mathcal{C} \to \mathcal{D}$ *to be* $\phi(F) = {}_{\mathcal{D}}\mathcal{D}_{F_{\mathcal{C}}} : \mathcal{C} \to \mathcal{D}$. *There is also a contravariant functor* $\hat{\phi} : \mathrm{Cat}^{\mathrm{op}} \to \mathbb{B}_R$ *that is again the identity on objects, and with* $\hat{\phi}(F) = {}_{\mathcal{C}^F}\mathcal{D}_{\mathcal{D}}$.
2. *Under these functors* ϕ *and* $\hat{\phi}$, *two functors* $F, G : \mathcal{C} \to \mathcal{D}$ *are sent to the same morphism in* $\mathbb{B}_R$ *if and only if* F *and* G *are naturally isomorphic.*

The Yoneda functor

We get functors $\mathrm{Cat} \to \mathbb{B}$ and $\mathrm{Cat}^{\mathrm{op}} \to \mathbb{B}$ that are the identity on objects, and that send a functor $F : \mathcal{C} \to \mathcal{D}$ to ${}_{\mathcal{D}}\mathcal{D}_{F_{\mathcal{C}}}$ in the covariant case and ${}_{\mathcal{C}^F}\mathcal{D}_{\mathcal{D}}$ in the contravariant case.

Proposition

Under these functors $\mathrm{Cat} \to \mathbb{B}$ and $\mathrm{Cat}^{\mathrm{op}} \to \mathbb{B}$, two functors $F, G : \mathcal{C} \to \mathcal{D}$ are sent to the same morphism in $\mathbb{B}$ if and only if F and G are naturally isomorphic.
If $\mathcal{D}$ is a poset then $F, G : \mathcal{C} \to \mathcal{D}$ are sent to the same morphism in $\mathbb{B}$ if and only if $F = G$.

The outer automorphism group of a category

Definition

$\operatorname{Out}\mathcal{C}$ is the group of self-equivalences of $\mathcal{C}$, up to natural isomorphism.

Corollary

Let $\mathcal{C}$ and $\mathcal{D}$ be finite categories.

1. *If $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories then they are isomorphic in the biset category.*
2. *The monoid homomorphism $\operatorname{End}_{\mathrm{Cat}}(\mathcal{C}) \to \operatorname{End}_{\mathbb{B}}(\mathcal{C})$ determined by the functor ϕ induces an injective group homomorphism*

$$\operatorname{Out}_{\mathrm{Cat}}(\mathcal{C}) \to \operatorname{Aut}_{\mathbb{B}}(\mathcal{C}).$$

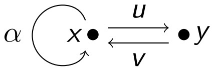
3. *For every biset functor F, the evaluation $F(\mathcal{C})$ has the structure of an $R\operatorname{Out}\mathcal{C}$-module.*

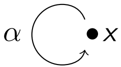
Idempotent completions

Theorem

Let R be a commutative ring with 1, and suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories. If the idempotent completions of $\mathcal{C}$ and $\mathcal{D}$ are equivalent then $\mathcal{C}$ and $\mathcal{D}$ are isomorphic in the biset category $\mathbb{B}_R$. It follows in this situation that if M is a biset functor then $M(\mathcal{C}) \cong M(\mathcal{D})$ and, in particular, the Burnside rings of $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

Application: the Burnside rings of the categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

$\mathcal{C} =$ 

$\mathcal{D} =$ 

with $uv = 1_y$ and $vu = \alpha$, $\alpha^2 = \alpha \neq 1_x$.

Factorizations as products of basic bisets

Bouc showed for groups that every indecomposable (G, J)-biset can be factorized

$$_G\Omega_J = {}_G G_H \circ {}_H Q_K \circ {}_K J_J$$

where $H \leq G$, $K \leq J$ and Q is an image of H and K.

The analogous statement is not true for bisets for categories in general.

However, every $(\mathcal{C}, \mathcal{D})$-biset $_\mathcal{C}\Omega_\mathcal{D}$ can be written as

$$_\mathcal{C}\Omega_\mathcal{D} = {}_\mathcal{C}\mathcal{E}_\mathcal{D} = {}_\mathcal{C}\mathcal{E}_\mathcal{E} \circ {}_\mathcal{E}\mathcal{E}_\mathcal{D}$$

where $\mathcal{E}$ is some category that has $\mathcal{C}$ and $\mathcal{D}$ as full subcategories. In our construction the category $\mathcal{E}$ has more morphisms than $\mathcal{C}$ or $\mathcal{D}$.

The cograph of a biset

Given a $(\mathcal{C}, \mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ we construct a category $\mathcal{E} = \mathrm{Cograph}(\Omega)$.
The objects of $\mathrm{Cograph}(\Omega)$ are $\mathrm{Ob}\,\mathcal{C} \sqcup \mathrm{Ob}\,\mathcal{D}$ and

$$\mathrm{Hom}_{\mathrm{Cat}(\Omega)}(x, y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x, y) & \text{if } x, y \in \mathcal{C}, \\ \mathrm{Hom}_{\mathcal{D}}(x, y) & \text{if } x, y \in \mathcal{D}, \\ \Omega(y, x) & \text{if } x \in \mathcal{D} \text{ and } y \in \mathcal{C}, \\ \emptyset & \text{if } x \in \mathcal{C} \text{ and } y \in \mathcal{D}. \end{cases}$$

Proposition

Let Ω be a $(\mathcal{C}, \mathcal{D})$-biset. Then $\mathcal{E} = \mathrm{Cograph}(\Omega)$ has $\mathcal{C}$ and $\mathcal{D}$ as full subcategories and $\Omega = {}_{\mathcal{C}}\mathcal{E}_{\mathcal{D}}$ as $(\mathcal{C}, \mathcal{D})$-bisets.