CENTRO DE CIENCIAS MATEMÁTICAS

Green Biset Functors

Topics in Representation Theory: Biset Functors

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Monoid object

Let $(\mathcal{M}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ be a monoidal category. A *monoid* in $\mathcal{M}$ is an object $m \in \mathcal{M}$ with morphisms

$$\mu : m \otimes m \longrightarrow m$$

and

$$e : \mathbf{1} \longrightarrow m$$

in $\mathcal{M}$ such that the following diagrams commute:

$$\begin{array}{ccccc} m \otimes (m \otimes m) & \xrightarrow{\mathrm{Id} \otimes \mu} & m \otimes m & & \\ \downarrow{\scriptstyle \alpha} & & & \searrow^{\mu} & \\ & & & & m \\ & & & \nearrow_{\mu} & \\ (m \otimes m) \otimes m & \xrightarrow{\mu \otimes \mathrm{Id}} & m \otimes m & & \end{array}
\qquad
\begin{array}{ccccc} \mathbf{1} \otimes m & \xrightarrow{e \otimes \mathrm{Id}} & m \otimes m & \xleftarrow{\mathrm{Id} \otimes m} & m \otimes \mathbf{1} \\ & \searrow_{\lambda} & \downarrow{\scriptstyle \mu} & \swarrow_{\rho} & \\ & & m & & \end{array}$$

Example

In (R-Mod, $\otimes$, R, . . .), monoid objects are the R-algebras.

Morphism of monoids

Let (m, μ, e) and (m', μ', e') be monoids in $\mathcal{M}$. A *monoid morphism* is a morphism $f : m \longrightarrow m'$ in $\mathcal{M}$ such that the following diagrams commute:

$$\begin{array}{ccc} m \otimes m & \xrightarrow{f \otimes f} & m' \otimes m' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ m & \xrightarrow[f]{\qquad} & m' \end{array} \qquad \begin{array}{ccc} & \mathbf{1} & \\ {\scriptstyle e}\swarrow & & \searrow{\scriptstyle e'} \\ m & \xrightarrow[f]{\qquad} & m \end{array}$$

Green biset functors

Definition

A *Green biset functor* (on $\mathcal{D}$, with values in R-Mod) is a monoid object in the monoidal category $(\mathcal{F}_{\mathcal{D},R}, \otimes, RB, \ldots)$.

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A *Green biset functor* (on $\mathcal{D}$, with values in R-Mod) is a monoid object in the monoidal category $(\mathcal{F}_{\mathcal{D},R}, \otimes, RB, \ldots)$.

This means that a Green biset functor is an object A of $\mathcal{F}_{\mathcal{D},R}$, together with maps of biset functors

$$\mu : A \otimes A \longrightarrow A$$

and

$$e : RB \longrightarrow A$$

such that the following diagrams commute:

$$\begin{array}{ccccc}
A \otimes (A \otimes A) & \xrightarrow{\mathrm{Id}_A \otimes \mu} & A \otimes A & & \\
 & & & \searrow^{\mu} & \\
\Big\downarrow \alpha & & & & A \\
 & & & \nearrow_{\mu} & \\
(A \otimes A) \otimes A & \xrightarrow[\mu \otimes \mathrm{Id}_A]{} & A \otimes A & &
\end{array} \qquad (1)$$

$$\begin{array}{ccccc}
RB \otimes A & \xrightarrow{e \otimes \mathrm{Id}_A} & A \otimes A & \xleftarrow{\mathrm{Id}_A \otimes e} & A \otimes RB \\
 & {}_{\lambda}\searrow & \Big\downarrow \mu & \swarrow_{\rho} & \\
 & & A & &
\end{array} \qquad (2)$$

where α, λ and ρ are the isomorphisms afforded by Proposition 8.4.6.

Equivalently, a Green biset functor is an object $A \in \mathcal{F}_{\mathcal{D},R}$ together with *bilinear products* $A(G) \times A(H) \longrightarrow A(G \times H)$, denoted by $(a, b) \longmapsto a \times b$, for objects G, H of $\mathcal{D}$, and an element $\varepsilon_A \in A(\mathbf{1})$, satisfying the following conditions:

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- (Associativity) If $\alpha_{G,H,K} : G \times (H \times K) \longrightarrow (G \times H) \times K$ is the canonical group isomorphism, then for any $a \in A(G)$, $b \in A(H)$, and $c \in A(K)$,

$$(a \times b) \times c = A\left(\mathrm{Iso}(\alpha_{G,H,K})\right)(a \times (b \times c)).$$

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- (Identity element) Let $\lambda_G : \mathbf{1} \times G \longrightarrow G$ and $\rho_G : G \times \mathbf{1} \longrightarrow G$ denote the canonical group isomorphisms. Then for any $a \in A(G)$,

$$A\left(\mathrm{Iso}(\lambda_G)\right)(\varepsilon_A \times a) = a = A\left(\mathrm{Iso}(\rho_G)\right)(a \times \varepsilon_A).$$

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- (Functoriality) If $\varphi : G \longrightarrow G'$ and $\psi : H \longrightarrow H'$ are morphisms in $R\mathcal{D}$, then for any $a \in A(G)$ and $b \in A(H)$,

$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b).$$

Proof

By the Universal Property of $\otimes$, we get

$$\mathrm{Hom}_{\mathcal{F}_{\mathcal{D},R}}(M \otimes N, P) \cong \mathrm{Hom}_{R\mathcal{D}\times R\mathcal{D}}(M(\_) \otimes N(\_), P(\_ \times \_))$$

for all $M, N, P \in \mathcal{F}_{\mathcal{D},R}$. If $M = N = P = A$ we have

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{F}_{\mathcal{D},R}}(A \otimes A, A) & \rightarrowtail\!\!\!\twoheadrightarrow & \mathrm{Hom}_{R\mathcal{D}\times R\mathcal{D}}(A(\_) \otimes A(\_), A(\_ \times \_)) \\ \mu & \longmapsto & \dot{\mu} \end{array}$$

Note that $\dot{\mu}$ is natural, so if $G, H, K, L \in R\mathcal{D}$, ${}_Hx_G$, ${}_Ly_K$ morphisms in $R\mathcal{D}$, $a \in A(G)$, and $b \in A(H)$, the following diagram commutes:

$$\begin{array}{ccc} A(G) \otimes_R A(K) & \xrightarrow{\dot{\mu}_{G,K}} & A(G \times K) \\ {\scriptstyle A(x)\otimes A(y)}\downarrow & & \downarrow{\scriptstyle A(x\times y)} \\ A(H) \otimes_R A(L) & \xrightarrow[\dot{\mu}_{H,L}]{} & A(H \times L) \end{array}$$

If we denote $\dot{\mu}_{G,K}(a \otimes b) := a \times b$, we'll have

$$\begin{array}{ccc} a \otimes b & \longmapsto & a \times b \\ \downarrow & & \downarrow \\ A(x)(a) \otimes A(y)(b) & \longmapsto & A(x)(a) \times A(y)(b) = A(x \times y)(a \times b) \end{array}$$

Therefore, we have functoriality.

Notice that $\dot{\mu}$ is a morphism in R-Mod, so it corresponds to a unique bilinear map, then we have the following commutative diagram:

$$\begin{array}{ccccc}
A(G) \times A(K) & \xrightarrow{\pi} & A(G) \otimes_R A(K) & \xrightarrow{\dot{\mu}_{G,K}} & A(G \times K) \\
{\scriptstyle A(x)\times A(y)} \downarrow & & {\scriptstyle A(x)\otimes A(y)} \downarrow & & \downarrow {\scriptstyle A(x\times y)} \\
A(H) \times A(L) & \xrightarrow[\pi]{} & A(H) \otimes_R A(L) & \xrightarrow[\dot{\mu}_{H,L}]{} & A(H \times L)
\end{array}$$

Let $\times := \dot{\mu} \circ \pi$, which is bilinear.

If we take $\varepsilon = [\mathbf{1}/\mathbf{1}] \in RB(\mathbf{1})$ and let x be a left G-set, notice that

$$x \cong x \times_{\mathbf{1}} \varepsilon.$$

Consider $e : RB \longrightarrow A$, so $e_{\mathbf{1}} : RB(\mathbf{1}) \cong R\varepsilon \longrightarrow A(\mathbf{1})$. We define

$$\varepsilon_A := e_{\mathbf{1}}(\varepsilon).$$

We have the following commutative diagram:

$$\begin{array}{ccc} RB \otimes A & \xrightarrow{e \otimes 1_A} & A \otimes A \\ & \overset{\iota}{\searrow} & \downarrow \mu \\ & & A \end{array}$$

By the bijective correspondence, we have

$$\begin{array}{ccc} RB(\_)\otimes_R A(\_) & \xrightarrow{e\otimes 1_{A(\_)}} & A(\_)\otimes_R A(\_) \\ & \overset{\check{\iota}}{\searrow} & \downarrow \mu \\ & & A(\_\times\_) \end{array}$$

where $\check{\iota} : RB(\_)\otimes_R A(\_) \longrightarrow A(\_\times\_)$ is such that $\forall\, G, H \in R\mathcal{D}$

$$\check{\iota}_{G,H} : RB(G)\otimes_R A(H) \longrightarrow A(G\times H)$$

$$x\otimes a \longmapsto A\left(({}_G x_1 \times Id_H)\circ \mathrm{Iso}_H^{1\times H}\right)(a)$$

This is a bilinear natural map.

With that definition, the following diagram commutes:

$$
\begin{array}{ccc}
RB(\mathbf{1}) \otimes_R A(G) & \xrightarrow{e_{\mathbf{1}} \otimes 1_{A(G)}} & A(\mathbf{1}) \otimes_R A(G) \\
\downarrow \beta_{\mathbf{1},G} & \searrow^{\check{\iota}_{\mathbf{1},G}} & \downarrow \dot{\mu}_{\mathbf{1},G} \\
A(G) & \xleftarrow[A(\mathrm{Iso}^{G}_{\mathbf{1}\times G})]{} & A(\mathbf{1} \times G)
\end{array}
$$

so

$$
\begin{array}{ccc}
\varepsilon \otimes a & \longmapsto & \varepsilon_A \otimes a \\
\downarrow & & \downarrow \\
a = A(\mathrm{Iso}^{G}_{\mathbf{1}\times G})(\varepsilon_A \times a) & \longleftarrow & \varepsilon_A \times a
\end{array}
$$

If $\varepsilon_A \in A(\mathbf{1})$ is such that for every $G \in R\mathcal{D}$ and for every $a \in A(G)$,

$$A(\mathrm{Iso}^G_{\mathbf{1}\times G})(\varepsilon_A \times a) = a = A(\mathrm{Iso}^G_{G\times \mathbf{1}})(a \times \varepsilon_A),$$

we have a bijective correspondence

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}_{\mathcal{D},R}}(RB, A) &\longrightarrow A(\mathbf{1}) \\ \eta &\longmapsto \eta_{\mathbf{1}}(\varepsilon) \\ e(a)_G : RB(G) \to A(G) \text{ s.t. } &\longleftarrow\!\shortmid\; a \\ x \mapsto A({}_G x_{\mathbf{1}})(a) & \end{aligned}$$

Setting $e_{\mathbf{1}}(\varepsilon) = \varepsilon_A$, we have a morphism $e : RB \longrightarrow A$.

Let $G, H \in R\mathcal{D}$, $x \in RB(G)$ and $a \in A(H)$. Note that

$$\begin{aligned} \dot{\mu}_{G,H}(e_G(x) \otimes a) &= \dot{\mu}_{G,H}\left(A(x)(\varepsilon_A) \otimes A(\mathrm{Id}_H)(a)\right) \\ &= A(x \times \mathrm{Id}_H)\dot{\mu}_{\mathbf{1},H}(\varepsilon_A \otimes a) \\ &= A(x \times \mathrm{Id}_H)(\varepsilon_A \times a) \\ &= A\left((x \times \mathrm{Id}_H) \circ \mathrm{Iso}_H^{\mathbf{1}\times H}\right)(a) \\ &= \check{\iota}_{G,H}(x \otimes a) \end{aligned}$$

Hence,

$$\begin{array}{ccc} RB(\_) \otimes_R A(\_) & \xrightarrow{e\otimes \mathrm{Id}_A} & A(\_) \otimes_R A(\_) \\ & \searrow^{\breve{\iota}} & \downarrow{\dot{\mu}} \\ & & A(\_ \times \_) \end{array}$$

commutes. By naturality of the correspondence, the diagram

$$\begin{array}{ccc} RB \otimes A & \xrightarrow{e\otimes \mathrm{Id}_A} & A \otimes A \\ & \searrow^{\iota} & \downarrow{\mu} \\ & & A \end{array}$$

commutes.

We have $\alpha : A \otimes (A \otimes A) \longrightarrow (A \otimes A) \otimes A$ that can be corresponded to

$$\dot{\alpha} : A(\_) \otimes_R (A(\_) \otimes_R A(\_)) \longrightarrow (A(\_) \otimes_R A(\_)) \otimes_R A(\_).$$

We can check that for every $M, N, P, T \in \mathcal{F}_{\mathcal{D},R}$,

$$\begin{aligned} \text{Hom}\,(M \otimes (N \otimes P), T) &\cong \text{Hom}\,(M(\_) \otimes_R (N(\_) \otimes_R P(\_))\,, T(\_ \times (\_ \times \_)))\ (*) \\ &\cong \text{Hom}\,((M(\_) \otimes_R N(\_)) \otimes_R P(\_), T((\_ \times \_) \times \_))\ (**) \end{aligned}$$

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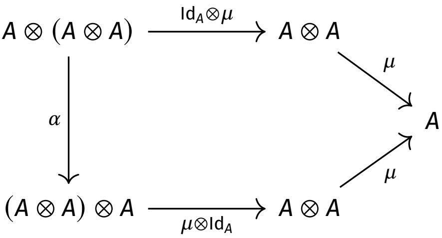
Let $\sigma_{G,H,K} := T\left(\mathrm{Iso}_{G\times(H\times K)}^{(G\times H)\times K}\right)$, then we have

$$\begin{aligned} (*) &\longrightarrow (**) \\ \eta &\longmapsto \sigma \circ \eta \circ \dot{\alpha} \end{aligned}$$

This is a natural isomorphism.

When $M = N = P = T = A$, we know that the following diagram commutes:



i.e., $\alpha^*(\mu \circ (\mu \otimes \mathrm{Id}_A)) = \mu \circ (\mathrm{Id}_A \otimes \mu)$.

Note that

$$\alpha^*(\mu \circ (\mu \otimes \mathrm{Id}_A)) \longmapsto \dot{\alpha}^*_{G,H,K}(\dot{\mu}_{G\times H,K} \circ (\dot{\mu}_{G,H} \otimes_R \mathrm{Id}_{A(K)}))$$

$$\mu \circ (\mathrm{Id}_A \otimes \mu) \longmapsto \dot{\mu}_{G,H\times K} \circ (\mathrm{Id}_{A(G)} \otimes \dot{\mu}_{H,K})$$

so the following hexagonal diagram commutes:

$$\begin{array}{ccc} A(G) \otimes_R (A(H) \otimes_R A(K)) & \xrightarrow{\dot{\alpha}_{G,H,K}} & (A(G) \otimes_R A(H)) \otimes_R A(K) \\ \Big\downarrow {\scriptstyle \mathrm{Id}_{A(G)} \otimes_R \dot{\mu}_{H,K}} & & \Big\downarrow {\scriptstyle \dot{\mu}_{G,H} \otimes_R \mathrm{Id}_{A(K)}} \\ A(G) \otimes_R A(H \times K) & & A(G \times H) \otimes_R A(K) \\ \Big\downarrow {\scriptstyle \dot{\mu}_{G,H\times K}} & & \Big\downarrow {\scriptstyle \dot{\mu}_{G\times H,K}} \\ A(G \times (H \times K)) & \xrightarrow[A\left(\mathrm{Iso}^{(G\times H)\times K}_{G\times(H\times K)}\right)]{} & A((G \times H) \times K) \end{array}$$

Morphism of Green biset functors

Definition

If (A, μ, e) and (A', μ', e') are Green biset functors on $\mathcal{D}$, with values in R-Mod, a *morphism of Green biset functors* from A to A' is a morphism $f : A \longrightarrow A'$ in $\mathcal{F}_{\mathcal{D},R}$ such that the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ \mu \downarrow & & \downarrow \mu' \\ A & \xrightarrow[f]{} & A' \end{array} \qquad \begin{array}{ccc} & RB & \\ {}^{e}\swarrow & & \searrow^{e'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

are commutative.

In other words, $f_{G\times H}(a \times b) = f_G(a) \times f_H(b)$ for any objects G, H of $\mathcal{D}$, and for any $a \in A(G)$ and $b \in A(H)$, and $f_{\mathbf{1}}(e_{\mathbf{1}}(\varepsilon)) = e'_{\mathbf{1}}(\varepsilon)$.

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Morphisms of Green biset functors can be composed, so Green biset functors on $\mathcal{D}$, with values in R-Mod, form a category $\text{Green}_{\mathcal{D},R}$ or $\mathcal{F}^{\mu}_{\mathcal{D},R}$.

Example 1

The Burnside functor B is a Green biset functor on $\mathcal{C}$, with values in $\mathbb{Z}$-mod, for the bilinear product

$$B(G) \times B(H) \longrightarrow B(G \times H)$$

for G, H finite groups, defined by

$$([X], [Y]) \longmapsto [X \times Y]$$

The identity element is $1 \in B(\mathbf{1}) \cong \mathbb{Z}$.

Example 2

Let $\mathbb{F}$ be a field of characteristic 0. The functor $R_{\mathbb{F}}$ sending a finite group G to $R_{\mathbb{F}}(G)$ the representation group of G over $\mathbb{F}$, is a Green biset functor: If $[V] \in R_{\mathbb{F}}(G)$ and $[W] \in R_{\mathbb{F}}(H)$, their product is defined as the class of the external tensor product $[V \boxtimes_{\mathbb{F}} W] \in R_{\mathbb{F}}(G \times H)$.

The identity element is the class of the trivial module $[\mathbb{F}] \in R_{\mathbb{F}}(\mathbf{1})$.

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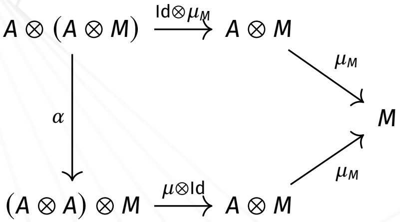
The linearization morphism $\chi_{\mathbb{F}} : B \longrightarrow R_{\mathbb{F}}$ is a morphism of Green biset functors.

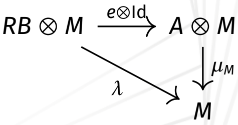
Module over a Green biset functor

Let A be a Green biset functor over $\mathcal{D}$, with values in R-Mod. A *left A-module* is an object M of $\mathcal{F}_{\mathcal{D},R}$, equipped with a morphism of biset functors

$$\mu_M : A \otimes M \longrightarrow M$$

such that the following diagrams commute:





Equivalently, for any objects G, H of $\mathcal{D}$, there are product maps $A(G) \times M(H) \longrightarrow M(G \times H)$, denoted by $(a, m) \longmapsto a \times m$, fulfilling the following conditions:

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Example

Let X be a fixed object of $\mathcal{D}$. The functor A_X obtained by the Yoneda-Dress construction is a left A-module, for the product map induced by the product in A, namely, since $A_X(H) = A(H \times X)$,

$$(a, m) \in A(G) \times A(H \times X) \mapsto a \times m \in A(G \times H \times X) = A_X(G \times H).$$

In particular, $A_{\mathbf{1}} \cong A$ is a left A-module.

Morphism of modules

If M and N are A-modules, then a *morphism of A-modules* from M to N is a morphism of biset functors $f : M \longrightarrow N$ such that the diagram

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mathrm{Id} \otimes f} & A \otimes N \\ {\scriptstyle \mu_M} \downarrow & & \downarrow {\scriptstyle \mu_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

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is commutative.
In other words, $f_{G \times H}(a \times m) = a \times f_H(m)$, for any objects G, H of $\mathcal{D}$, any $a \in A(G)$ and any $m \in M(H)$.

Morphisms of A-modules can be composed, and A-modules form a category, denoted by A-Mod.

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Remark

The category A-Mod is an abelian category.

Submodules of an A-module

Let A be a Green biset functor and M be an A-module. An *A-submodule* of M is a subfunctor N of M such that the image of the morphism

$$A(G) \times N(H) \longrightarrow M(G \times H)$$

is contained in $N(G \times H)$.

Ideals of a Green biset functor

Let A be a Green biset functor on $\mathcal{D}$, with values in R-Mod. A *left ideal* of A is an A-submodule of the left A-module A.

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In other words, it is a biset subfunctor I of A such that the image of the morphism

$$A(G) \times I(H) \longrightarrow A(G \times H)$$

is contained in $I(G \times H)$ for any objects G and H of $\mathcal{D}$.

A two sided ideal I of A is a biset subfunctor such that

$$A(G) \times I(H) \times A(K) \subseteq I(G \times H \times K)$$

for any objects G, H, K of $\mathcal{D}$.

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for any objects G, H, K of $\mathcal{D}$.

When I is a two sided ideal of A, the quotient A/I is a Green biset functor. The projection morphism from A to A/I is a morphism of Green functors.

Simple Green biset functors

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Let G be an object of $\mathcal{D}$. We define $\mathcal{I}(kR_{\mathbb{F},G})$ as the set of ideals of $kR_{\mathbb{F},G}$ and $I(k, \mathbb{F}, G)$ as the set of primitive idempotents of $kR_{\mathbb{F}}(G)$.

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In particular, if $G = 1$, $|\mathcal{I}(kR_{\mathbb{F}})| = 2$, so $kR_{\mathbb{F}}$ is a simple Green biset functor.

Thank you!