The Dade Group
12.1 & 12.2

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Let p be a prime number, P be a p - group, and k be a field of characteristic p.

Definition

A permutation kP-module M is a kP-module that admits a P-invariant k-basis X. Equivalently, M is isomorphic to kX for some P-set X.

Note: Throughout this chapter, we will assume that all permutation kP-modules are finitely generated or equivalently finite-dimensional over k.

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Now, kP - modules of the form kP/Q where Q is a subgroup of P are indecomposable with vertex Q.

Therefore any direct summand of a permutation kP-module is a permutation kP-module.

Also, by the Krull-Schmidt theorem, Y is another P-invariant k-bases of M then the P-sets X and Y are isomorphic.

Permutation Modules

Remark

Let $V \cong kX$ and $W \cong kY$ be permutation kP-modules for some P-sets X and Y. Then,

$$V \oplus W \cong k(X \sqcup Y)$$

and

$$V \otimes_k W \cong k(X \times Y)$$

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Remark

Let P and Q be p-groups, and U be a finite (Q, P)-biset.
If V is a permutation kP-module, with P-invariant k basis X, then

$$kU \otimes_{kP} V \cong k(U \times_P X)$$

and is hence a permutation kQ-module.

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If V is a permutation kP-module, with P-invariant k basis X, then

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and is hence a permutation kQ-module.

Therefore, the class of permutation modules is closed under the usual biset operations on modules, namely induction, restriction, inflation, and deflation.

The Brauer Quotient

Let k be a field, and P be a group.

Definition

If V is a kP-module, and $Q \leqslant P$, the Brauer quotient $V[Q]$ of V at Q is the $kN_P(Q)/Q$-module defined by

$$V[Q] = V^Q / \sum_{S<Q} tr_S^Q V^S,$$

where $tr_S^Q : V^S \to V^Q$ is the trace map $v \mapsto \sum_{x \in Q/S} xv$.

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The correspondence $V \mapsto V[Q]$ is a functor from kP-mod to $kN_P(Q)/Q$-mod, denoted by Br_Q^P (or Br_Q is P is clear from context).

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Remark

When $V \cong kX$ is a permutation kP-module, where X is some P-set, the image of the set X^Q in $V[Q]$ is a k-basis of $V[Q]$ and $V[Q]$ is a permutation $kN_P(Q)/Q$-module.

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Defintion

A P-algebra A is called primitive if the identity element 1_A is a primitive idempotent of the algebra A^P.

Permutation Algebras

Remark

If A is a P-algebra over k, and if $Q \leqslant P$, then $\sum_{S<Q} tr_S^Q A^S$ is a two-sided ideal of A^Q, so $A[Q]$ is a $N_P(Q)/Q$-algebra over k.

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Remark

If A is a permutation P-algebra over k with P-invariant basis X, and if $Q \trianglelefteq P$, then the natural bijection

$$(X^Q)^{P/Q} \cong X^P$$

induces an algebra isomorphism $A[Q][P/Q] \to A[P]$.

12.2 Endo-Permutation Modules

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In this definition, the action of P on $End_k(M)$ is given by

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for $x \in P, f \in End_k(M)$, and $m \in M$.

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Example

All permutation kP-modules are endo-permutation modules.

Examples

Example

If M is an endo-permutation kP-module, then its k-dual $M^* = Hom_k(M,k)$ is also an endo-permutation kP-module since $End_k(M^*) \cong M \otimes_k M^* \cong End_k(M)$ as M is finite dimensional over k.

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Let X be a nonempty finite P-set. Let $\varepsilon_X : kX \to k$ be the augmentation map, defined by $\varepsilon_X(x) = 1$, for $x \in X$.

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Therefore, there is a short exact sequence:

$$0 \longrightarrow \Omega_X(k) \longrightarrow kX \longrightarrow k \longrightarrow 0.$$

$\Omega_X(k)$ is an endo-permutation module.

Examples

Example

If M and N are endo-permutation kP-modules, then so is $M \otimes_k N$, since

$$(M \otimes_k N) \otimes (M \otimes N)^* \cong (M \otimes_k M^*) \otimes_k (N \otimes_k N^*)$$

is a permutation kP-module.

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Example

If M and N are endo-permutation kP-modules, then

$$(M \oplus N) \otimes_k (M \oplus N)^* \cong (M \otimes_k M^*) \oplus (M \otimes_k N^*) \oplus (N \otimes_k M^*) \oplus (N \otimes_k N^*)$$

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Therefore, if $M \oplus N$ is an endo-permutation kP-module, then $M \otimes_k N^*$ is a permutation kP-module.

Conversely, if $M \otimes_k N^*$ is a permutation kP-module, then $N \otimes_k M^* \cong (M \otimes_k N^*)^*$ is a permutation module, and $M \oplus N$ is an endo-permutation kP-module.

Definitions

Let k be a field of characteristic $p > 0$ and P be a p-group.

Definition

Two endo-permutation kP-modules M and N are said to be compatible (which is denoted by $M \sim N$) if $M \oplus N$ is an endo-permutation kP-module, or, equivalently, if $M \otimes_k N^*$ is a permutation kP-module.

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Definition

An endo-permutation kP-module M is said to be capped if it admits an indecomposable summand with vertex P.

Lemma 12.2.6

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Let k be a field of characteristic $p > 0$, P be a p-group, and M be an endo-permutation kP-module.

The following are equivalent:

1) The module M is capped.
2) The Brauer quotient $End_k(M)[P]$ is non-zero.
3) The trivial module k appears as a direct summand of the kP-module $End_k(M) \cong M \otimes_k M^*$.

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Now for each $Q \leqslant P$, kP/Q is a permutation module with vertex Q.
Recall if N is an indecomposable permutation kP-module with vertex Q and $R \leqslant P$, then $N[R] \neq 0$ if $R \leqslant_P Q$ and $N[R] = 0$ otherwise.

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Recall if N is an indecomposable permutation kP-module with vertex Q and $R \leqslant P$, then $N[R] \neq 0$ if $R \leqslant_P Q$ and $N[R] = 0$ otherwise. Therefore, $End_k(M)[P] \neq 0$ iff k is an indecomposable summand of $End_k(M)$

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Let k be a field of characteristic $p > 0$ and P be a p-group.
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Proof: The first claim follows from Lemma 12.2.6 since

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Proof: The first claim follows from Lemma 12.2.6 since

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Suppose that M and N are both capped endo-permutation kP-modules. Now,

$$(M \otimes_k N) \otimes_k (M \otimes_k N)^* \cong (M \otimes_k M^*) \otimes_k (N \otimes_k N^*)$$

has a direct summand isomorphic to k since both $M \otimes_k M^*$ and $N \otimes_k N^*$ have one.

12.2.8 Theorem & Definition [Dade]

Let k be a field of characteristic $p > 0$ and P be p-group.

1) The relation $\sim$ is an equivalence relation on the class of capped endo-permutation kP-modules. Let $D_k(P)$ denote the set of equivalence classes for this relation.

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2) Let M and N be capped endo-permutation kP-modules. Then $M \sim N$ if and only if $M^* \sim N^*$.
3) If M, N, M', and N' are capped endo-permutation kP-modules such that $M \sim N$ and $M' \sim N'$, then $M \otimes_k M' \sim N \otimes_k N'$.

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2) Let M and N be capped endo-permutation kP-modules. Then $M \sim N$ if and only if $M^* \sim N^*$.
3) If M, N, M', and N' are capped endo-permutation kP-modules such that $M \sim N$ and $M' \sim N'$, then $M \otimes_k M' \sim N \otimes_k N'$.
4) The tensor product of modules induces an addition on $D_k(P)$, defined by

$$[M] + [N] = [M \otimes_k N],$$

where $[M]$ denotes the equivalence class of the capped endo-permutation kP-module M. Then $D_k(P)$ is an Abelian group for this addition law, called the Dade group of P over k. The zero element of $D_k(P)$ is the class $[k]$ of the trivial module, and the opposite of the class of M is the class of M^*.

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Assertion 2: Follows from

$$(M \oplus N)^* \cong M^* \oplus N^*$$

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Assertion 3: Follows from

$$(M \otimes_k N) \otimes_k (M' \otimes_k N')^* \cong (M \otimes_k M'^*) \otimes_k (N \otimes_k N'^*)$$

and the tensor product of two permutation kP-modules is a permutation kP-module.

Proof of 12.2.8 continued

Assertion 4: The addition on $D_k(P)$ is well-defined by Assertion 3 and it is straightforward to show that the addition is commutative and associative.

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$[k]$ is the zero element of $D_k(P)$ since $M \otimes_k k \cong M$ for each kP-module M.

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Therefore, for each capped endo-permutation module M, the module $M \otimes_k M^*$ is in the equivalence class $[k]$ and hence $[M] + [M^*] = [k]$.

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1) If V is a capped indecomposable endo-permutation kP-module, then $V \sim M$ if and only if V is isomorphic to a direct summand of M.
2) In particular, if V and W are indecomposable summands of M with vertex P, then $V \cong W$.

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This means that $D_k(P)$ is the set of isomorphism classes of capped indecomposable endo-permutation kP-modules.