Problems on Green Biset Functors and the Dade Group

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R = routine
C = Challenging
O = Open research problem

1 Green biset functors

1. (R) Let M be a biset functor over a commutative ring R. Show that there exists an isomorphism

$$\mathcal{H}(RB, M) \cong M$$

of biset functors over R which is natural in M.

3. (C) Let k be a field of characteristic p. Find a subcategory $\mathcal{D}$ of the biset category $\mathcal{C}$ that is as large as possible so that the Grothendieck group $R(kG)$ of finitely generated kG-modules (with respect to short exact sequences) is a biset functor. Is it a Green biset functor on $\mathcal{D}$

2. (R) Let A be a Green biset functor over a commutative ring R. Show that, for any positive integer n, there exists a Green biset functor $\mathrm{Mat}_n(A)$ whose evaluation at a finite group G is equal to $\mathrm{Mat}_n(A(G))$.

3. (a) (R) Show that the Burnside ring functor $G \mapsto B(G)$ has the structure of a Green biset functor.

(b) (R) Show that the character ring functor $G \mapsto R(G)$ has the structure of a Green biset functor.

(c) (R) Show that the maps $B(G) \to R(G)$ sending the class $[X] \in B(G)$ of a finite G-set X to the character of the permutation $\mathbb{C}G$-module $\mathbb{C}X$ define a morphism of Green biset functors.

4. (R) Let A be a Green biset functor over a commutative ring R. Show that there exists a unique morphisms $\eta\colon RB \to A$ of Green biset functors over R.

2 The Dade Group

1. Let p be a prime, k a field of characteristic p, and $P = \langle x \rangle$ a cyclic group of order p^n.

(a) (R) Show that the three k-algebras kP, $k[T]/(T^{p^n}-1)$, and $k[T]/((T-1)^{p^n})$ are isomorphic.

(b) (R) Show that each submodule of the regular kP-module is of the form $M_i = (x-1)^i kP$ for $i = 0, 1, 2, \ldots, p^n$ and that $\dim_k M_i = p^n - i$.

(c) (R) Show that every indecomposable kP-module is of the form $U_i := kP/M_i$ for some $i = 0, \ldots, p^n$.

(d) (C) Decompose $U_i \otimes_k U_j$ into a direct sum of indecomposable modules.

(e) (R) For which $i = 0, \ldots, p^n$ is U_i a permutation kP-module?

(f) (C) For which $i = 0, \ldots, p^n$ is U_i an endo-permutation module?

(g) (R with f) For which $i = 0, \ldots, p^n$ is U_i a capped endo-permutation module?

(h) (R with g) Describe the Dade group $D_k(P)$.

2. (Following Alperin) Let p be a prime, k a field of characteristic p, P a p-group and X be a finite P-set. Consider the kP-module homomorphism

$$\varepsilon\colon kX \to k \tag{1}$$

which sends each element x of X to 1. The goal is to show that $\Omega(X) := \ker(\varepsilon)$ is an endo-permutation kP-module.

(a) (R) Consider (1) as a chain complex C with kX in degree 0. Show that the k-dual of C is isomorphic to the chain complex

$$\eta\colon k \to kX \tag{2}$$

where $\sigma(1)$ is the sum of all elements in X. Show that $H_0(C) = \Omega(X)$ and $H_0(C^*) = \Omega(X)^*$.

(b) (C) Consider the tensor product complex $C \otimes_k C^*$. Show that $C \otimes_k C^*$ is contractible as chain complex of kP-modules and that

$$H_0(C \otimes_k C^*) \cong \Omega(X) \otimes_k \Omega(X)^* .$$

(c) (R) Derive from (b) that $\Omega(X)$ is an endo-permutation kP-module.

3. (C) Let p be a prime, k a field of characteristic p, a P a finite p-group. Further, suppose that

$$0 \to L \to P \to M \to 0$$

is a short exact sequence of finitely generated kP-modules with P a projective module. Show that M is a capped endo-permutation kP-module if and only if L is.

4. (C) Let p be a prime, k a field of characteristic p, G a finite group and M a finitely generated kG-module. Recall that the *Brauer quotient* (or *Brauer construction*) $M[P]$ of M at a p-subgroup P of G is defined as the $k(N_G(P)/P)$-module

$$M[P] := M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q) .$$

Suppose that $M = kX$ for a finite G-set X. Show that the composition

$$k[X^P] \subseteq (kX)^P \to (kX)[P]$$

is an isomorphism of $k(N_G(P)/P)$-modules.

5. (O) Let $\mathcal{O}$ be a complete discrete valuation ring with residue field of characteristic p, as for example the ring of p-adic integers $\mathbb{Z}_p$, and let P be a p-group.

A finitely generated $\mathcal{O}G$-module L, which is free as an $\mathcal{O}$-module, is called an *endo-monomial* $\mathcal{O}P$-module if $\mathrm{End}_{\mathcal{O}}(L)$ is a monomial $\mathcal{O}P$-module, i.e., if it is isomorphic to a direct sum of $\mathcal{O}P$-modules of the form $\mathrm{Ind}_Q^P \mathcal{O}_\varphi$, where $Q \leq P$, $\varphi \in \mathrm{Hom}(Q, \mathcal{O}^\times)$ and $\mathcal{O}_\varphi$ is the kQ-module whose underlying $\mathcal{O}$-module is just $\mathcal{O}$ and on which Q acts via φ. In his PhD thesis, Robert

Hartmann developed the theory of endo-monomial $\mathcal{O}P$-modules. However, the following question remained open.

Question: Does there exist an indecomposable endo-monomial $\mathcal{O}P$-module L with vertex P which is not an endo-permutation, i.e., such that $\mathrm{End}_{\mathcal{O}}(L)$ is not a permutation $\mathcal{O}P$-module. The answer is "no" if P is abelian. See [Hartmann: Endo-monomial modules over p-groups and their classification in the abelian case; J. Algebra 274 (2004), 564—586].

6. (R) Let p be a prime and let k be field of characteristic p. Further, let P and Q be p-groups $f\colon U \to V$ a morphism in the category of finite (Q, P)-bisets. Show that f induces a natural transformation between the two functors T_U and T_V from $\underline{\mathrm{perm}}_k(P)$ to $\underline{\mathrm{perm}}_k(Q)$.