CENTRO DE CIENCIAS MATEMÁTICAS

The third definition of Green biset functor

Topics in Representation Theory: Biset Functors

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The definition of Green biset functor [Bouc

Definition (Bouc)

Let $A \in \mathcal{F}_{D,R}$ is a Green biset functor if it is equipped with bilinear products $A(G) \times A(H) \longrightarrow A(G \times H)$ denoted by $(a, b) \longmapsto a \times b$, for groups G, H in $\mathcal{D}$, and an element $\xi_A \in A(1)$, satisfying the following conditions:

1. Associativity. Let G, H and K be groups in $\mathcal{D}$. If we consider the canonical isomorphism from $G \times (H \times K)$ to $(G \times H) \times K$, then for any $a \in A(G)$, $b \in A(H)$ and $c \in A(K)$

$$(a \times b) \times c = A(Iso_{G\times(H\times K)}^{(G\times H)\times K})(a \times (b \times c)).$$

2. Identity element. Let G be a group in $\mathcal{D}$ and consider the canonical isomorphisms $1 \times G \longrightarrow G$ and $G \times 1 \longrightarrow G$. Then for any $a \in A(G)$

$$a = A(Iso_{1\times G}^{G}(\xi_A \times a) = A(Iso_{G\times 1}^{G}(a \times \xi_A)$$

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Definition

3. Functoriality. If $\varphi : G \longrightarrow G'$ and $\psi : H \longrightarrow H'$ are morphisms in $R\mathcal{D}$, then for any $a \in A(G)$ and $b \in A(H)$

$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b).$$

The definition of Green biset functor [Ron

Definition (Romero Thesis)

Definition 2. Let $A \in \mathcal{F}_{D,R}$, is a Green biset functor if $A(H)$ is an R-algebra with unity , for each group H in $\mathcal{D}$, and satisfies the following. If K and G are groups in $\mathcal{D}$ and $\varphi : K \longrightarrow G$ is a group homomorphism, then:

1. For the (K, G)-biset G, which we denote by ${}_{K\varphi}G_G$, the morphism $A({}_{K\varphi}G_G)$ is a ring homomorphism.
2. For the (G, K)-biset G, denoted by ${}_GG_{\varphi K}$, the morphism $A({}_GG_{\varphi K})$ satisfies the Frobenius identities for all $b \in A(G)$ and $a \in A(K)$,

$$A({}_GG_{\varphi K})(a) \cdot b = A({}_GG_{\varphi K})(a \cdot A({}_{K\varphi}G_G)(b))$$
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where $\cdot$ denotes the ring product on $A(G)$, resp. $A(K)$.

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Equivalent

Lemma (Lemma 4.2.3 in Romero Thesis [2])

The two previous definitions are equivalent.

Demostración.

First, we star by Definition Bouc, the ring structure of $A(H)$ is given by

$$a \cdot b = A(Iso^{H}_{\Delta(H)} \circ Res^{H\times H}_{\Delta(H)})(a \times b),$$

for a and b in $A(H)$, with the unity given by $A(Inf^{H}_{1})(\xi)$.

Conversely, starting by Definition Romero, the product of $A(G) \times A(H) \longrightarrow A(G \times H)$ is given by

$$a \times b = A({}_{G\times H^{p_1}} G_G)(a) \cdot A({}_{G\times H^{p_2}} H_H)(b)$$

for $a \in A(G)$ and $b \in A(H)$, with the identity element given by the unity of $A(1)$.

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Proof

First, we will show that Definition Romero implies Definition Bouc. To do this, we first need to define the functor $\times$. Let G and H be elements of $\mathcal{D}$. We define the function as follows:

$$\begin{aligned} \times : A(G) \times A(H) &\longrightarrow A(G \times H) \\ (a, b) &\longmapsto A({}_{G\times H^{p_1}}G_G)(a) \cdot A({}_{G\times H^{p_2}}H_H)(b). \end{aligned}$$

where p_1 and p_2 represent the first and second projections, respectively. Now, we will demonstrate that $\times$ satisfies the following properties:

- R-bilinearity: For all G and H in $\mathcal{D}$, the function

  $$\times : A(G) \times A(G) \longrightarrow A(G \times H)$$

  is R-linear, since $A({}_{G\times H^{p_2}}H_H)$ and $A({}_{G\times H^{p_1}}G_G)$ are morphisms of R-algebras.

- Functoriality: By the bilinearity of $\times$ and Bouc's decomposition it suffices to demonstrate naturality for the elementary biset.
  Let G, H, and L be elements of $\mathcal{D}$, and let ${}_{L}X_{G}$ be an elementary (L, G)-biset. We want to prove that the following diagram commutes:

$$\begin{array}{ccc} A(G) \times A(H) & \xrightarrow{\times} & A(G \times H) \\ {\scriptstyle (A({}_{L}X_{G}), Id_{H})} \downarrow & & \downarrow {\scriptstyle A({}_{L\times H}X \times H_{G\times H})} \\ A(L) \times A(H) & \xrightarrow{\times} & A(L \times H). \end{array}$$

  In other words, we need to show that:

$$A\left({}_{L\times H}{}^{P_1}L_L \circ_L X_G\right)(a) \cdot A({}_{L\times H}{}^{P_2}H_H)(b)$$

  is equal to

$$A\left({}_{L\times H}X \times H_{G\times H}\right)\left(A({}_{G\times H}{}^{P_1}G_G)(a) \cdot A({}_{G\times H}{}^{P_2}H_H)(b)\right).$$

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Note that the elementary bisets are of the form ${}_{L\,\varphi}G_G$ with $\varphi : L \longrightarrow G$ being a group morphism, or of the form ${}_L L_{\rho\, G}$ with $\rho : G \longrightarrow L$ being a group morphism. Let's consider two cases:

- Case 1: ${}_L X_G = {}_{L^{\varphi}}\ G_G$.
  By hypothesis, $A\left({}_{L\times H^{(\varphi,1)}} G\times H_{G\times H}\right)$ is an R-algebra morphism, then:

$$A\left({}_{L\times H^{(\varphi,1)}} G\times H_{G\times H}\right)\left(A({}_{G\times H^{P_1}} G_G)(a)\cdot A({}_{G\times H^{P_2}} H_H)(b)\right)$$

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  Now, we have:

$$\begin{aligned}{}_{L\times H^{(\varphi,1)}} G\times H_{G\times H} \circ_{G\times H^{P_1}}\ G_G &\cong {}_{L\times H^{P_1\circ(\varphi,1)}}\ G_G\\ &\cong {}_{L\times H^{\varphi\circ P_1}}\ G_G,\end{aligned}$$

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  Now, we have:

$$\begin{aligned}{}_{L\times H^{(\varphi,1)}} G \times H_{G\times H} \circ_{G\times H^{P_1}}\ G_G &\cong {}_{L\times H^{P_1\circ(\varphi,1)}}\ G_G \\ &\cong {}_{L\times H^{\varphi\circ P_1}}\ G_G,\end{aligned}$$

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- Case 1: ${}_L X_G =_{L^{\varphi}} G_G$.
  By hypothesis, $A\left({}_{L\times H^{(\varphi,1)}} G \times H_{G\times H}\right)$ is an R-algebra morphism, then:

$$A\left({}_{L\times H^{(\varphi,1)}} G \times H_{G\times H}\right)\left(A({}_{G\times H^{P_1}} G_G)(a) \cdot A({}_{G\times H^{P_2}} H_H)(b)\right)$$

is equal to

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Now, we have:

$$\begin{aligned} {}_{L\times H^{(\varphi,1)}} G \times H_{G\times H} \circ_{G\times H^{P_1}} G_G &\cong_{L\times H^{P_1\circ(\varphi,1)}} G_G \\ &\cong_{L\times H^{\varphi\circ P_1}} G_G, \end{aligned}$$

and

$${}_{L\times H^{(\varphi,1)}} G \times H_{G\times H} \circ_{G\times H^{P_2}} H_H \cong_{L\times H^{P_2}} H_H$$

Therefore,

$$A\left({}_{L\times H}{}^{(\varphi,1)}G\times H_{G\times H}\right)\left(A({}_{G\times H}{}^{P_1}G_G)(a)\cdot A({}_{G\times H}{}^{P_2}H_H)(b)\right)$$

is equal to

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Therefore the diagram commutes.

- Case 2: ${}_LX_G =_L L_{\rho G}$ We have the following diagram:

$$\begin{array}{ccc} A(G)\times A(H) & \xrightarrow{\times} & A(G\times H) \\ {\scriptstyle (A({}_L L_{\rho G}),Id_H)}\big\downarrow & & \big\downarrow{\scriptstyle A({}_{L\times H}L\times H_{(1,\rho)\,G\times H})} \\ A(L)\times A(H) & \xrightarrow{\times} & A(L\times H). \end{array}$$

  Let $a\in A(G)$ and $b\in A(H)$. By following the diagram, we obtain:

$$\left(A\left({}_L L_{\rho G}\right)(a)\right)\times b = A\left({}_{L\times H}{}^{P_1}L_L\circ_L L_{\rho G}\right)(a)\cdot A\left({}_{L\times H}{}^{P_2}H_H\right)(b),$$

  and we have that:

$$A\left({}_{L\times H}L\times H_{(\rho,1)}{}_{G\times H}\right)(a\times b)$$

is equal to

$$A\left({}_{L\times H}L\times H_{(\rho,1)}{}_{G\times H}\right)\left(A\left({}_{G\times H^{P_1}}G_G\right)(a)\cdot A\left({}_{G\times H^{P_2}}H_H\right)(b)\right).$$

One has

$${}_{L\times H^{P_2}}H_H \cong_{G\times H^{(\rho,1)}} L\times H_{L\times H}\circ_{L\times H^{P_2}} H_H.$$

We conclude that

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Using Frobenius,

$$A\left({}_{L\times H}L\times H_{(\rho,1)}{}_{G\times H}\circ_{G\times H^{P_1}} G_G\right)(a)\cdot A\left({}_{L\times H^{P_2}}H_H\right)(b),$$

note that,

$${}_{L\times H}L\times H_{(\rho,1)}{}_{G\times H}\circ_{G\times H^{P_1}} G_G \quad \text{and} \quad {}_{L\times H^{P_1}}L_L\circ_L L_{\rho G}$$

are isomorphic as $((L\times H), G)$-biset. Therefore, the above diagram commutes.

$$A\left({}_{L\times H}L\times H_{(\rho,1)}{}_{G\times H}\right)\left(A\left(G\times H^{P_1}G_G\right)(a)\cdot A\left(G\times H^{(\rho,1)}L\times H_{L\times H}\circ_{L\times H^{P_2}}H_H\right)(b)\right).$$

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identity element: Let ξ the multiplicative identity of $A(1)$. Let $G \in \mathcal{D}_0$ and $a \in A(G)$. Notice that:

$$\begin{aligned}
A({}_G G_{G\times 1})(a \times \xi) &= A({}_G G_{G\times 1})(A({}_{G\times 1} G_G)(a) \cdot A({}_G 1_1)(1)) \\
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&= A({}_G G \times 1_G)(a) \cdot A({}_G G \times 1_1)(\xi) \\
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where ξ_G is the multiplicative identity of $A(G)$.

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Modules

Definition

Let $A \in \mathcal{F}^{\mu}_{D,R}$. A biset functor M is an A-module, if $M(H)$ is a $A(H)$-module that satisfies the following. Let K and G are groups in $\mathcal{D}$ and $\varphi : K \longrightarrow G$ is a group homomorphism, then:

1. For all $a \in A(G)$ and $m \in M(G)$ one has

$$M({}_{K\varphi}G_G)(a \cdot m) = A({}_{K\varphi}G_G)(a) \cdot M({}_{K\varphi}G_G)(m).$$

2. For the (G, K)-biset G, denoted by ${}_GG_{\varphi G}$, the morphism $A({}_GG_{\varphi G})$ satisfies the Frobenius identities for all $a \in A(K)$, $b \in A(G)$, $m \in M(G)$ and $n \in M(K)$,

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Burnside Functor

The Burnside group $B_{\mathcal{D}}(G)$ is the Grothendieck group of the category $\mathcal{D}$ with respect to the disjoint unions of morphisms. The R-Mod $RB_{\mathcal{D}}(G) := R \otimes_{\mathbb{Z}} B_{\mathcal{D}}(G)$ is a ring with unity. The multiplication operation is defined as follows:

$$[X] \cdot [Y] := [X \times Y]$$

where X and Y are G-set. The unity element is the G-set $[\cdot]$.

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The Fibered Burnside Functor

Let A be a multiplicative abelian group,

Definition

Let X be a set, we call X an A-fibered G-set if X is an $A \times G$-set such that the action of A is free with A-orbits are finitely.

We denote by ${}_G set^A$ the category of A-fibered G-sets. Here the morphisms are given by $A \times G$-equivariant functions. The operation of disjoint union of sets induces a coproduct on ${}_G set^A$

- $B^A(G)$ the Grothendieck group of this category with respect to disjoint unions.

- Let be a (G, H)-biset. we define the map

$$RB^A(U) : RB^A(H) \longrightarrow RB^A(G)$$
$$[X] \longmapsto [U \otimes_{AH} X].$$

where $[U \otimes_{AH} X]$ are the elements of $[U \circ_H X]$ such that the action of A is free

The group $RB^A(G)$ has a structure ring via :

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The slice Burnside functor

Definition

Let G finite group. The category of morphisms of G-sets. to be denoted G-Mor, consist

- Obj(G-Mor)= the morphisms of G-sets.
- Let $f : A \longrightarrow B$ and $g : A' \longrightarrow B'$ be morphisms of G-sets.

$$Hom_{G-\text{Mor}}(A \xrightarrow{f} B, A' \xrightarrow{g} B') := \{(h,k) \mid h,k \in (G\text{-Mor})_0 \text{ and } (1)\}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h\downarrow & & \downarrow k \\ A' & \xrightarrow{g} & B'. \end{array} \tag{1}$$

- The composition is the composition of functions
- $(1,1)$ is the identity of $A \xrightarrow{f} B$.

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Let $A \xrightarrow{f} B$ and $A' \xrightarrow{g} B'$ elements of G-Mor. We define the disjoint union of these morphisms as follows:

$$A \sqcup A' \xrightarrow{f \sqcup f'} B \sqcup B$$
$$x \longmapsto f \sqcup f'(x)$$

where

$$f \sqcup f'(x) = \begin{cases} f(x) \text{ if } x \in A \\ f'(x) \text{ if } x \in A'. \end{cases}$$

$\sqcup$ is a cooproduct of G-Mor.

The slice Burnside group of G, denotade by $\Xi(G)$, is the Grothendieck group of the category G-Mor with respect to disjoint unions of morphisms.

Definition

Let U be a (G, H)-biset, we define the map

$$\Xi(U) : \Xi(G) \longrightarrow \Xi(H)$$
$$(X \xrightarrow{f} Y) \longmapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y),$$

where $U \times_G X$ and $U \times_G Y$ have the natural action of H-sets coming from the action of H on U.

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where $U \times_G X$ and $U \times_G Y$ have the natural action of H-sets coming from the action of H on U.

Let $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} W$ be elements of G-Mor. We define

$$X \times Z \xrightarrow{f \times g} Y \times W \tag{2}$$

$$(x, y) \longmapsto (f(x), g(x)) \tag{3}$$

The group $\Xi(G)$ has struture of ring via

$$[X \xrightarrow{f} Y] \cdot [Z \xrightarrow{g} W] = [X \times Z \xrightarrow{f \times g} Y \times W]$$

The element idenity is $\{\cdot\} \xrightarrow{1} \{\cdot\}$.

The shifted Functor

Let K be a finite group. The Green biset functor A over $R\mathcal{D}$ can be shifted by K. This gives a new Green biset functor, A_K, defined for a finite group G by

$$A_K(G) = A(G \times K).$$

For finite groups G and H and a finite (H, G)-biset U, the map

$$A_K(U) : A_K(G) \longrightarrow A_K(H)$$

is the map $A(U \times K)$, where $U \times K$ is viewed as a $(H \times K, G \times K)$-biset in the obvious way.
Moreover, for a finite group G , $A_K(G) = A(G \times K)$ is a R-algebra whit unity.

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Thank you!