Planar Graphs

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Abstract

The main topic of this note is to explain the motivation behind Kuratowski's theorem for planar graphs.

1 Introduction.

Planar graphs are important in areas like network design, circuit layouts, and map-making, where minimizing crossings can make things simpler and easier to understand. By studying them, we gain insight not just into how to draw graphs more clearly, but also into deeper ideas in topology, graph theory, and real-world problem solving.

Definition 1.1. A graph $\langle G, E\rangle$ is *planar* if it can be embedded into the plane in such a way that its edges intersect only at their endpoints.

Graphs can have different visual representations depending on how their vertices and edges are arranged in the plane. A graph that appears non-planar in one drawing may actually be planar if redrawn without any edge crossings. This means that planarity is a property of the graph itself—not just of a particular drawing—so it's important to consider whether a graph can be drawn without crossings, rather than relying solely on its appearance. For example, consider K_4, the complete graph with 4 vertices, in the following two presentations.

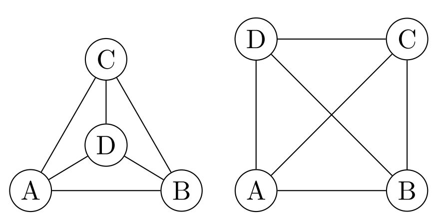

Figure 1: Two different presentation of K_4.

In Figure 1, two representations of the same graph are shown. Note that in the one on the left, edges only intersect at their respective endpoints, and the one on the right, the edge $\overline{AC}$ intersects the edge $\overline{BD}$ exactly in the middle.

[*] wikipedia.org

2 Euler's formula.

Not all graphs are planar; one of the earliest proofs that tests the planarity of a graph is Euler's formula for planar graphs.

Theorem 2.1 (Euler's formula for planar graphs)**.** *Let $\langle G, E \rangle$ be a planar graph, let v the number of vertices, e the number of edges and f be the number of faces, including the exterior face, then*

$$v - e + f = 2.$$

Consider the graph K_5 –the complete graph with 5 vertices–. Using Euler's formula, we can easily show that K_5 is not planar.

Theorem 2.2. *The complete graph with 5 vertices is not planar.*

Proof. Notice that, for K_5, the number of vertices v is and the number of edges e is 10. If K_5 was to be planar, then, by Euler's formula, such embedding must have $f = 2 - v + e = 7$ faces. Now, notice that every face must be bounded by at least 3 edges, and each edge borders at most 2 faces. Therefore $2e \geq 3f$. However this means that $20 \geq 21$, which is a contradiction. $\square$

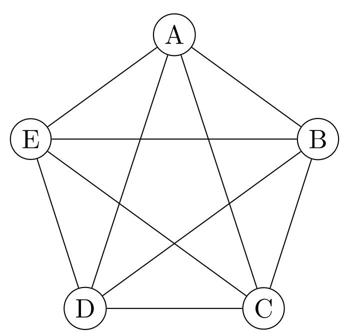

Figure 2: The complete graph with 5 vertices, K_5.

The reader may find a similar proof for $K_{3,3}$, the complete bipartite graph in which each piece has 3 vertices. It turns out that these two graphs are a basis for finite non-planar graphs.

3 Kuratowski's theorem

Kuratowski's theorem is a mathematical forbidden graph characterization of planar graphs, named after Kazimierz Kuratowski.

Theorem 3.1 (Kuratowski's theorem)**.** *A finite graph G is planar if and only if it is not possible to subdivide the edges of K_5 or $K_{3,3}$, and then possibly add additional edges and vertices, to form a graph isomorphic to G.*

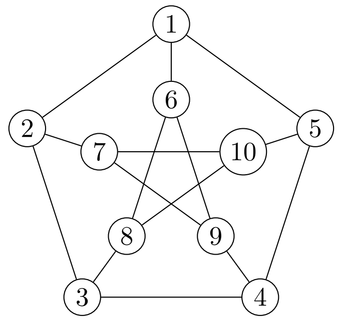

Figure 3: Petersen graph

Kazimierz Kuratowski published his theorem in 1930 (see [1]). The theorem was independently proved by Orrin Frink and Paul Smith, also in 1930, but their proof was never published. The special case of cubic planar graphs was also independently proved by Karl Menger in 1930. Since then, several new proofs of the theorem have been discovered.

In the Soviet Union, Kuratowski's theorem was known as either the Pontryagin–Kuratowski theorem or the Kuratowski–Pontryagin theorem, as the theorem was reportedly proved independently by Lev Pontryagin around 1927. However, as Pontryagin never published his proof, this usage has not spread to other places.

References

[1] K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. **15** (1930), 271–283.