The four color theorem

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Abstract

The four color theorem states that no more than four colours are required to color the regions of any map so that no two adjacent regions have the same color.

The four color theorem was proven in [1] after many false proofs and counterexamples. It was the first major theorem to be proved using a computer. Initially, this proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand. Since then the proof has gained wide acceptance, although some doubters remain.

1 Basic Notions

Definition 1. *The* chromatic number of a graph G *(denoted by* $\chi(G)$*) is the smallest amount of colors required to color the vertices of* G *such that no adjacent vertices have the same color. A graph is* planar *if it can be embedded into* $\mathbb{R}^2$.

Two important examples of non-planar graphs are the following:

- The complete graph with 5 vertices, denoted by K_5.
- The utility graph, denoted by $K_{3,3}$ is the smallest triangle-free graph in which every vertex has exactly three neighbours.

One can easily show that $\chi(K_5) = 5$ and $\chi(K_{3,3}) = 2$. The following result can be consulted in [2].

Theorem 2 (Kuratowski, Wagner)**.** *A finite graph is planar if and only if does not have* K_5 *or* $K_{3,3}$ *as a minor.*

2 Regions of a map

One way to code a map divided in many regions is with graphs: Vertices can be used to code the regions of the map and edges can be used to code adjacence between regions. Intuitively, coding a map using this method will give you a planar graph.

Theorem 3 ([1])**.** *The chromatic number of a planar graph is at most 4.*

*All the information here is taken from wikipedia.org

Since 1890, it was known that the chromatic number of a planar graph was at most 5. It was not until 1976 that, with the power of computers, a proof for Theorem 3 arised. An easy generalization to infinite graphs can be stated in the following way.

Theorem 4. *The chromatic number of a graph such that every finite subgraph is planar is at most 4.*

References

[1] K. Appel and W. Haken. Every planar map is four colorable. *Bull. Amer. Math. Soc.*, 82(5):711–712, 1976.

[2] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.