Groups

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March 26, 2025

Abstract

In this work we introduce the notion of a group and prove some basic properties about them.

1 Introduction

The modern concept of an abstract group developed out of several fields of mathematics. The original motivation for group theory was the quest fo solutions of polynomial equations of degree higher than 4. Geometry was a second field in which groups were used systematically, especially symmetry groups as a part of Felix Klein's 1872 Erlangen program. The third field contributiong to group theory was number theory. Certain abelian group structures had been used implicitly in Carl Friedrich Gauss's number-theoretical work Disquisitiones Arithmeticae.

Definition. *A* group *is a set G together with a binary operation $\cdot : G \times G \to G$ that combines two elements $a, b \in G$ to form an element of G denoted by $a \cdot b$ such that the following requirements, known as group axioms, are satisfied:*

- ***Associativity.*** *For all $a, b, c \in G$, one has $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,*
- ***Identity element.*** *There exists an element $e \in G$ (denoted as the* identity element of the group*) such that, for every $a \in G$, one has $e \cdot a = a = a \cdot e$,*
- ***Inverse element.*** *For each $a \in G$, there exists an element $b \in G$ such that $a \cdot b = b \cdot a = e$, where e is the identity element.*

Examples

Some examples include basic structures as

- the set of reals $\mathbb{R}$ with the addition $+$,
- the rationals $\mathbb{Q} \setminus \{0\}$ with the product $\cdot$,
- the set of all bijective functions from a set X to itself with the composition,
- the set of invertible matrices in $\mathbb{R}^{n \times n}$ with the product of matrices,
- the set $\{0, 1\}$ with the operation defined in the following table is a group.

*as usual, the primary source of this is wikipedia.org

+	0	1
0	0	1
1	1	0

Table 1: The operation on $\mathbb{Z}_2$

2 Basic properties

Basic facts about groups can be obtained using only the group axioms.

Proposition (Uniqueness of the identity)**.** *There is a unique* $e \in G$ *such that for all* $a \in G$*,* $a \cdot e = e \cdot a = a$*.*

Proof. If $e, e' \in G$ have such property, then $e = e \cdot e' = e'$. □

Proposition (Uniqueness of the inverses)**.** *For all* $a \in G$ *there is an unique element* $a' \in G$ *such that* $a \cdot a' = a' \cdot a = e$*.*

Proof. Let $a \in G$ and assume that $a', a'' \in G$ are inverses of a. Then $a' = a' \cdot (a \cdot a'') = (a' \cdot a) \cdot a'' = a''$. □

Groups can be found in many different branches of mathematics, computer science and physics.

3 Classification of simple finite groups

A *simple group* is a nontrivial group whose only normal subgroups are the trivial group and itself. The following theorem is left as an easy warm up exercise to the reader.

Theorem. *Every finite simple group is isomorphic to one of the following:*

- *a cyclic group of prime order,*
- *an alternating group of degree at least 5,*
- *a group of Lie type,*
- *one of the 27 sporadic groups.*