# CANJAR FILTERS

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ABSTRACT. If $\mathcal{F}$ is a filter on $\omega$, we say that $\mathcal{F}$ is *Canjar* if the corresponding Mathias forcing does not add a dominating real. We prove that any Borel Canjar filter is $F_\sigma$, this solves a problem of Hrušák and Minami. We give several examples of Canjar and non-Canjar filters, in particular, we construct a MAD family such that the corresponding Mathias forcing adds a dominating real. This answers a question of Brendle. Then we prove that in all the "classical" models of ZFC there are MAD families whose Mathias forcing does not add a dominating real. We also study ideals generated by branches, and we uncover a close relation between Canjar ideals and the selection principle $S_{fin}(\Omega, \Omega)$ on subsets of the Cantor space.

## 1. Introduction

Given a filter $\mathcal{F}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if it adds a pseudointersection to $\mathcal{F}$. There are two classical partial orders for diagonalizing a filter $\mathcal{F}$, the *Laver forcing* relative to $\mathcal{F}$, denoted by $\mathbb{L}(\mathcal{F})$, which consists of all trees of height $\omega$ that have a stem and above it the set of successors of every node is a member of $\mathcal{F}$, and there is also the *Mathias forcing* relative to $\mathcal{F}$, which is defined as $\mathbb{M}(\mathcal{F}) = \left\{(s, A) \mid s \in [\omega]^{<\omega} \wedge A \in \mathcal{F}\right\}$, the order is given by $(s, A) \leq (z, B)$ whenever $z$ is an initial segment of $s$, $s - z \subseteq B$ and $A \subseteq B$. These partial orders have many properties in common, but in general they are distinct forcing notions; for example, it is easy to see that $\mathbb{L}(\mathcal{F})$ always adds a dominating real, while this is not necessarily the case for $\mathbb{M}(\mathcal{F})$. It is folklore knowledge that if $\mathcal{U}$ is a Ramsey ultrafilter, then $\mathbb{M}(\mathcal{U})$ is equivalent to $\mathbb{L}(\mathcal{U})$, hence adds a dominating real (this has been implicitly proved in [14]). On the other hand, under $\mathfrak{d} = \mathfrak{c}$, Canjar constructed an ultrafilter whose Mathias forcing does not add a dominating real (see [5]). We call such type of filters *Canjar filters*. We say that an ideal $\mathcal{I}$ is a *Canjar ideal* if its dual filter $\mathcal{I}^* = \{\omega - X \mid X \in \mathcal{I}\}$ is a Canjar filter. Canjar filters have been investigated in [9] and [3], this paper is a continuation of that line of research.

In [9] Hrušák and Minami found a combinatorial reformulation of being Canjar. If $W$ is a countable set, we denote by $fin(W)$ as the set of all non empty finite subsets of $W$. If $\mathcal{I}$ is an ideal on $W$, we define the ideal $\mathcal{I}^{<\omega}$ as the set of all $A \subseteq fin(W)$ such that there is $Y \in \mathcal{I}$ with the property that $a \cap Y \neq \emptyset$ for all $a \in A$. We will write $fin$ instead of $fin(W)$ when is clear of context. Recall

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that $\mathcal{I}$ is a *$P^+$-ideal* if every decreasing sequence of positive sets has a positive pseudointersection. The characterization of Hrušák and Minami is the following.

**Proposition 1** ([9]). *$\mathcal{I}$ is a Canjar ideal if and only if $\mathcal{I}^{<\omega}$ is a $P^+$-ideal.*

In [4] Brendle showed that every $F_\sigma$ ideal is a Canjar ideal. It was asked by Hrušák and Minami if every Borel Canjar ideal must be $F_\sigma$ and one of the main results of this article is to answer this question positively. In order to achieve this, we will extend a characterization of Canjar ultrafilters by Blass, Hrušák and Verner in [3].

We say that a MAD family is *Canjar* if the ideal generated by it is Canjar. In [4] Brendle showed that under $\mathfrak{b} = \mathfrak{c}$ there is a non Canjar MAD family, and asked if it is possible to construct one in ZFC. We show that this is indeed the case. We then turn our attention to constructing a Canjar MAD family, and we show that in many of the "classical" models of ZFC there is one. We do not know if this is true in general.

We also study ideals generated by branches, and we show that there is a connection between Canjar ideals and selection principles on the Cantor space.[1]

Using the previous ideas, in [7] we gave alternative proofs of the consistency of $\mathfrak{b} < \mathfrak{a}$ and $\mathfrak{b} < \mathfrak{s}$ (which were proved by Shelah [20]).

Our notation is standard and follows mostly [1], by $\mathcal{I}^+$ we will denote the set of subsets of $\omega$ that are not in $\mathcal{I}$ and are called the *positive sets with respect to $\mathcal{I}$* or $\mathcal{I}$-positive sets. Whenever $a, b$ are two sets, $a - b$ will denote the set theoretic difference of $a$ and $b$. The definition of the basic cardinal invariants such as $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{r}$, non$(\mathcal{M})$ and cov$(\mathcal{M})$ may be consulted in [2].

## 2. Canjar ideals

Given $A \subseteq fin$, we denote by $\mathcal{C}(A)$ as the set of all $X \subseteq \omega$ such that $a \cap X \neq \emptyset$ for all $a \in A$. We may identify $\wp(\omega)$ [2] with $2^\omega$, which is homeomorphic to the Cantor set endowed with the product topology. In this way, we can talk about topological properties (like compact, $F_\sigma$ or Borel) of families of subsets of $\omega$. The next lemma is easy and its proof is left to the reader.

**Lemma 1.**
(1) *If $A \subseteq fin$, then $\mathcal{C}(A)$ is compact, and if $A \in (\mathcal{I}^{<\omega})^+$ then $\mathcal{C}(A) \subseteq \mathcal{I}^+$.*
(2) *If $\mathcal{C} \subseteq \wp(\omega)$ is compact and $X \subseteq \omega$ intersects every element of $\mathcal{C}$, then there is $F \in [X]^{<\omega}$ such that $F$ intersects every element of $\mathcal{C}$.*
(3) *If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are compact, then $\mathcal{D} = \{A_1 \cap \ldots \cap A_n \mid A_i \in \mathcal{C}_i\}$ is also compact.*

A slightly less trivial lemma is the following.

[1]This connection has recently been further studied in [6].
[2]We are using $\wp(Z)$ to denote the power set of $Z$.

**Lemma 2.** *Let $\mathcal{F}$ be a filter , $X \subseteq fin$ be such that $\mathcal{C}(X) \subseteq \mathcal{F}$ and $\mathcal{D}$ compact with $\mathcal{D} \subseteq \mathcal{F}$. Then, for every $n \in \omega$ there is $S \in [X]^{<\omega}$ such that if $A_0, \ldots, A_n \in \mathcal{C}(S)$ and $F \in \mathcal{D}$ then $A_0 \cap \ldots \cap A_n \cap F \neq \emptyset$.*

*Proof.* Given $s \in X$ define $K(s)$ as the set of all $(A_0, \ldots, A_n) \in \mathcal{C}(s)^{n+1}$ with the property that there is $F \in \mathcal{D}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$, this is a compact set by the previous lemma. Note that if $(A_0, \ldots, A_n) \in \bigcap\limits_{s \in X} K(s)$ then $A_0, \ldots, A_n \in \mathcal{C}(X) \subseteq \mathcal{F}$ and there would be $F \in \mathcal{D} \subseteq \mathcal{F}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$ which is clearly a contradiction. Since the $K(s)$ are compact, there must be $S \in [F]^{<\omega}$ such that $\bigcap\limits_{s \in S} K(s) = \emptyset$. It is easy to see that this is the $S$ we are looking for. $\square$

Now we prove the theorem of Canjar using the characterization of Hrušák and Minami. This is an elaboration of the proof that there is a $P$-point under $\mathfrak{d} = \mathfrak{c}$ (see [2]).

**Proposition 2** ([5]). *If $\mathfrak{d} = \mathfrak{c}$ then there is a Canjar ultrafilter.*

*Proof.* Let $\langle \overline{X}_\alpha \mid \alpha \in \mathfrak{c} \rangle$ be an enumeration of all decreasing sequences of subsets of $[\omega]^{<\omega}$. Recursively, we will construct a continuous increasing sequence of filters $\langle \mathcal{U}_\alpha \mid \alpha \in \mathfrak{c} \rangle$ such that for all $\alpha < \mathfrak{c}$,

(1) $\mathcal{U}_\alpha$ is the union of less than $\mathfrak{d}$ compact sets,
(2) either $\overline{X}_\alpha$ is not a sequence of $\mathcal{U}^{<\omega}$ positive sets or it has a pseudointersection $P$ such that $\mathcal{C}(P) \subseteq \mathcal{U}_{\alpha+1}$.

We begin by setting $\mathcal{U}_0$ to be the cofinite subsets of $\omega$ and we take the union at limit stages. Assume that we have already defined $\mathcal{U}_\alpha$, we will see how to define $\mathcal{U}_{\alpha+1}$. In case $\overline{X}_\alpha = \langle X_n \mid n \in \omega \rangle$ is not a sequence of $\mathcal{U}^{<\omega}$ positive sets we just do $\mathcal{U}_{\alpha+1} = \mathcal{U}_\alpha$. Now assume that each $X_n \in \mathcal{U}^+$, which implies that $\mathcal{C}(X_n) \subseteq \mathcal{U}^+$, we will find a compact set $\mathcal{D}$ such that $\mathcal{U}_\alpha \cup \mathcal{D}$ generates a filter, and this will be $\mathcal{U}_{\alpha+1}$, by point 3 of lemma 1, $\mathcal{U}_{\alpha+1}$ will be generated by less than $\mathfrak{c}$ compact sets.

In case there is $n \in \omega$ such that $\mathcal{C}(X_n)$ is not contained in $\mathcal{U}_\alpha$, we choose $Y \in \mathcal{C}(X_n) - \mathcal{U}_\alpha$ and define $\mathcal{D} = \{\omega - Y\}$. In this way, $\overline{X}_\alpha$ is no longer a sequence of positive sets. So assume $\mathcal{C}(X_n) \subseteq \mathcal{U}_\alpha$ for each $n \in \omega$. Let $\mathcal{U}_\alpha = \bigcup\limits_{\beta \in \kappa} \mathcal{C}_\beta$ where $\mathcal{C}_\beta$ is compact and $\kappa$ is less than $\mathfrak{d}$. By the previous lemma, for every $\beta < \kappa$ we can define a function $f_\beta : \omega \longrightarrow \omega$ such that for every $n \in \omega$ there is $S \in [X_n]^{<\omega}$ with $S \subseteq \wp(f_\beta(n))$ such that if $A_0, \ldots, A_{n+1} \in \mathcal{C}(S)$ and $F \in \mathcal{C}_\beta$ then $A_0 \cap \ldots \cap A_{n+1} \cap F \neq \emptyset$. Since $\{f_\beta \mid \beta < \kappa\}$ is not dominating, there is $g$ that is not dominated by any of the $f_\beta$. Let $P = \bigcup\limits_{n \in \omega} \wp(g(n)) \cap X_n$. It is clear that $P$ is a pseudointersection. Now we claim that $\mathcal{U}_\alpha \cup \mathcal{C}(P)$ generates a filter. For this, let $F \in \mathcal{U}_\alpha$ and $B_0, \ldots, B_n \in \mathcal{C}(P)$. We must show $B_0 \cap \ldots \cap B_n \cap F \neq \emptyset$. Pick $\beta < \kappa$ such that $F \in \mathcal{C}_\beta$, and since $g \nleq^* f_\beta$, there is $m > n$ such that $g(m) > f_\beta(m)$. By the construction, then there is $S \in [X_m]^{<\omega}$ with $S \subseteq \wp(f_\beta(m)) \subseteq \wp(g(m))$ such that if $A_0, \ldots, A_{n+1} \in \mathcal{C}(S)$ then $A_0 \cap \ldots \cap A_{n+1} \cap F \neq \emptyset$, but clearly $B_0, \ldots, B_n \in \mathcal{C}(S)$ so we are done.

Finally, let $\mathcal{U} = \bigcup\limits_{\alpha < \mathfrak{c}} \mathcal{U}_\alpha$. Then, by the construction, $\mathcal{U}$ is a Canjar ultrafilter. $\square$

In [12] Laflamme introduced the following notion for ultrafilters.

**Definition 1.** *We say that $\mathcal{I}$ is a* strong $P^+$-ideal *if for every increasing sequence $\langle \mathcal{C}_n \mid n \in \omega \rangle$ of compact sets with $\mathcal{C}_n \subseteq \mathcal{I}^+$, there is an interval partition $\mathcal{P} = \langle P_n \mid n \in \omega \rangle$ such that if $\langle X_n \mid n \in \omega \rangle$ is a sequence with $X_n \in \mathcal{C}_n$ for all $n \in \omega$ then $\bigcup\limits_{n\in\omega} (X_n \cap P_n) \in \mathcal{I}^+$.*

Laflamme noted without a proof that Canjar ultrafilters were strong $P^+$-filters and asked if these two notions were equivalent. This was answered positively by Blass, Hrušák and Verner in [3]. We will now extend their result to the general case.

**Definition 2.** *We say that $\mathcal{I}$ is a* coherent strong $P^+$-ideal *if for every increasing sequence $\langle \mathcal{C}_n \mid n \in \omega \rangle$ of compact sets with $\mathcal{C}_n \subseteq \mathcal{I}^+$, there is an interval partition $\mathcal{P} = \langle P_n \mid n \in \omega \rangle$ such that if $\langle X_n \mid n \in \omega \rangle$ is a sequence with the following "coherence property" for $\mathcal{P}$,*

(1) *$X_n \in \mathcal{C}_n$ for all $n \in \omega$,*
(2) *if $n < m$ then $X_m \cap P_n \subseteq X_n \cap P_n$.*

*then $\bigcup\limits_{n\in\omega} X_n \cap P_n \in \mathcal{I}^+$.*

Note that the coherence property is satisfied if the $\langle X_n \mid n \in \omega \rangle$ is decreasing, as well as, when $\mathcal{I}$ is the dual of an ultrafilter. We will now prove that an ideal is Canjar if and only if it satisfies the coherent strong $P^+$-ideal property.

**Proposition 3** ([3] for ultrafilters)**.** *An ideal $\mathcal{I}$ is Canjar if and only if $\mathcal{I}$ is a coherent strong $P^+$-ideal.*

*Proof.* First assume that $\mathcal{I}$ is a Canjar ideal. Let $\langle \mathcal{C}_n \mid n \in \omega \rangle$ be an increasing sequence of compact sets with $\mathcal{C}_n \subseteq \mathcal{I}^+$. For every $n \in \omega$ define $A_n$ as the set of all $a \in [\omega]^{<\omega}$ such that if $X \in \mathcal{C}_n$ then $a \cap X \neq \emptyset$. We will see that $A_n \in (\mathcal{I}^{<\omega})^+$. Let $B \in \mathcal{I}$. We must find an element of $A_n$ that is disjoint from $B$. For every $y \notin B$ define $V_y = \{X \in \mathcal{C}_n \mid y \in X\}$. Since $\mathcal{C}_n \subseteq \mathcal{I}^+$, we conclude that $\langle V_y \mid y \notin B \rangle$ is an open cover of $\mathcal{C}_n$ so there is a finite $a \subseteq \omega - B$ such that $\mathcal{C}_n = \bigcup\limits_{y\in a} V_y$. Therefore $a \in A_n$ and $a \cap B = \emptyset$.

In this way $\langle A_n \mid n \in \omega \rangle$ is a decreasing sequence of positive sets and since $\mathcal{I}$ is Canjar, there is $A \subseteq^* A_n$ with $A \in (\mathcal{I}^{<\omega})^+$. We may as well assume that $A \subseteq A_0$. Define an interval partition $\mathcal{P} = \langle P_n \mid n \in \omega \rangle$ in such a way that for all $n \in \omega$ if $a \in A - A_n$ then $a \subseteq \bigcup\limits_{i<n} P_i$. We will see that this is the partition we are looking for. Let $\langle X_n \mid n \in \omega \rangle$ be a sequence with the coherence property for $\mathcal{P}$. We will show that $X = \bigcup\limits_{n\in\omega} X_n \cap P_n \in \mathcal{I}^+$. It is enough to show that $X$ intersects every element of $A$ (because if $X \in \mathcal{I}$ then $A$ will be in $\mathcal{I}^{<\omega}$ which is a contradiction). Let $a \in A$ and define $n =\max\{m \mid a \cap \bigcup\limits_{i\leq m} P_i \neq \emptyset\}$. Since $a \not\subseteq \bigcup\limits_{i<n} P_i$, $a$ must be in $A_n$, hence $a \cap X_n \neq \emptyset$. By the coherence property, we know that $\bigcup\limits_{i\leq n} X_n \cap P_i \subseteq \bigcup\limits_{i\leq n} X_i \cap P_i \subseteq X$ so $a \cap X \neq \emptyset$.

Now assume that $\mathcal{I}$ is a coherent strong $P^+$-ideal. We shall show that $\mathcal{I}^{<\omega}$ is a $P^+$-ideal. Let $\langle A_n \mid n \in \omega \rangle \subseteq (\mathcal{I}^{<\omega})^+$ be a decreasing sequence. We must find a positive pseudointersection. For every $n \in \omega$ define $\mathcal{C}_n = \{X \subseteq \omega \mid \forall a \in A_n \, (a \cap X \neq \emptyset)\}$. Since $\mathcal{C}_n$ is an intersection of compact sets, it is compact and it is easy to see that $\mathcal{C}_n \subseteq \mathcal{I}^+$. Let $\mathcal{P} = \langle P_n \mid n \in \omega \rangle$ be an interval partition witnessing that $\mathcal{I}$ is a coherent strong $P^+$-ideal. Call $E_n = \bigcup\limits_{i \leq n} P_i$ and define $A = \bigcup\limits_{n \in \omega} (A_n \cap \wp(E_n))$. Clearly $A \subseteq^* A_n$ for every $n \in \omega$ so it remains to show that $A$ is positive. Assume this is not the case, so there is $B \in \mathcal{I}$ that intersects every element of $A$. Define $X_n = (B \cap E_n) \cup (\omega - E_n)$ and note that $X_n \in \mathcal{C}_n$ and $\langle X_n \mid n \in \omega \rangle$ satisfies the coherence property for $\mathcal{P}$. In this way $B = \bigcup\limits_{n \in \omega} (X_n \cap P_n) \in \mathcal{I}^+$ which is a contradiction. □

As an application, we will show that all $F_\sigma$ ideals are Canjar.

**Proposition 4** ([4])**.** *Every $F_\sigma$ ideal is a Canjar ideal.*

*Proof.* Let $\mathcal{I}$ be an $F_\sigma$ ideal. We will show that it is a coherent strong $P^+$-ideal. By a theorem of Mazur (see [16]) there is a lower semicontinuous submeasure $\varphi : \wp(\omega) \longrightarrow [0, \infty]$ [3] such that $\mathcal{I} = \{A \mid \varphi(A) < \omega\}$.

Let $\langle \mathcal{C}_n \mid n \in \omega \rangle$ be an increasing sequence of compact positive sets. Since each $\mathcal{C}_n$ is compact, it is easy to recursively construct an interval partition $\langle P_n \mid n \in \omega \rangle$ such that $\varphi(P_n \cap Y) > n$ for each $Y \in \mathcal{C}_n$. In this way, it is clear that $\bigcup\limits_{n \in \omega} X_n \cap P_n \in \mathcal{I}^+$, whenever $X_n \in \mathcal{C}_n$. □

Actually, in [4] Brendle showed that if $\mathcal{I}$ is the union of less than $\mathfrak{d}$ compact sets, then $\mathcal{I}$ is Canjar. In [9] it was asked if every Borel Canjar ideal is $F_\sigma$, in the next section we will prove that this is indeed the case.

## 3. Borel Canjar ideals

Recall another notion introduced by Laflamme and Leary in [13]. We say that a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ is an *$\mathcal{I}^+$-tree of finite sets* if for every $t \in T$, there is $X_t \in \mathcal{I}^+$ such that $suc_T(t) = [X_t]^{<\omega}$.

**Definition 3.** *We say that $\mathcal{I}$ is a $P^+$*(tree)-ideal *if for every $\mathcal{I}^+$-tree of finite sets $T$, there is $b \in [T]$ such that $\bigcup\limits_{n \in \omega} b(n) \in \mathcal{I}^+$.*

We will show that Canjar ideals are $P^+(tree)$,

**Proposition 5.** *If $\mathcal{I}$ is Canjar, then $\mathcal{I}$ is $P^+(tree)$.*

*Proof.* Let $T \subseteq ([\omega]^{<\omega})^{<\omega}$ be an $\mathcal{I}^+$-tree of finite sets. For convenience, denote by $\omega^{\nearrow\omega}$ the set of all increasing finite sequences of natural numbers. We define a subtree $T' = \{t_s \mid s \in \omega^{\nearrow\omega}\} \subseteq T$ in the following way,

[3] We say that $\varphi : \wp(\omega) \longrightarrow [0, \infty]$ is a *lower semicontinuous submeasure* if $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$, and $\varphi(A) = lim_{n \longrightarrow \infty} \varphi(A \cap n)$

(1) $t_\emptyset = \emptyset$,
(2) $t_{\langle n \rangle} = X_\emptyset \cap [0, n)$ for every $n \in \omega$,
(3) $t_{\langle n_0,\ldots,n_{m+1} \rangle} = X_{t_{\langle n_0 \cdots n_m \rangle}} \cap [n_m, n_{m+1})$.

Let $Y_\emptyset = X_\emptyset$. If $s^\frown \langle n \rangle \in \omega^{\nearrow\omega}$ define $Y_{s^\frown \langle n \rangle} = (Y_s \cap n) \cup (X_{s^\frown \langle n \rangle} - n)$. Call $\mathcal{C}_n = \{Y_s \mid s \in \omega^{\nearrow\omega} \wedge |s| \leq n\}$. It is easy to see that $\langle \mathcal{C}_n \mid n \in \omega \rangle$ is an increasing sequence of compact positive sets (for example, one may note that if $Y \in \overline{\mathcal{C}_{n+1}}$ then either $Y \in \{Y_s \mid |s| = n+1\}$ or it is in the closure of $\mathcal{C}_n$). Find $\mathcal{P} = \langle P_n \mid n \in \omega \rangle$ an interval partition that witnesses that $\mathcal{I}$ is Canjar. Define the function $l : \omega \longrightarrow \omega$ where $l(n)$ is the right end-point of $P_n$ and consider the branch $b = \langle t_{l \restriction n} \rangle$. We will see that $\bigcup\limits_{n\in\omega} t_{l\restriction n} \in \mathcal{I}^+$. Note that $Y_{l\restriction n} \in \mathcal{C}_n$ and $\langle Y_{l\restriction n} \mid n \in \omega \rangle$ satisfie the coherence property for $\mathcal{P}$ so $\bigcup\limits_{n\in\omega} Y_{l\restriction n} \cap P_n \in \mathcal{I}^+$ but $\bigcup\limits_{n\in\omega} Y_{l\restriction n} \cap P_n = \bigcup\limits_{n\in\omega} t_{l\restriction n}$ which is what we were looking for. □

However, being Canjar is a stronger notion than being $P^+(tree)$, we will later see an example of an ideal that is $P^+(tree)$ but not Canjar.

**Theorem 1.** *If $\mathcal{I}$ is a Borel ideal, then the following are equivalent,*

(1) *$\mathcal{I}$ is Canjar,*
(2) *$\mathcal{I}$ is $F_\sigma$,*
(3) *$\mathcal{I}$ is $P^+(tree)$.*

*Proof.* The equivalence between 2 and 3 was proved by Hrušák and Meza in [8] and the other equivalence follows from the previous results. □

In [5] Canjar proved that if a forcing notion adds a dominating real, then it must have size at least $\mathfrak{d}$. It follows that every ideal generated by less than $\mathfrak{d}$ sets is Canjar, since its Mathias forcing has a dense set of size less than $\mathfrak{d}$. With this observation and the previous theorem, we can conclude the following result of Veličković and Louveau,

**Corollary 1** (Veličković, Louveau see [15])**.** *If $\mathcal{I}$ is a Borel non $F_\sigma$-ideal then $cof(\mathcal{I}) \geq \mathfrak{d}$.*

Note that there are Borel (non $F_\sigma$) ideals of cofinality $\mathfrak{d}$, one example is $FIN \times FIN$ which is the ideal in $\omega \times \omega$ generated by all columns $C_n = \{(n, m) \mid m \in \omega\}$ and all $A \subseteq \omega \times \omega$ such that $A$ intersects every $C_n$ in a finite set.

## 4. Canjar MAD families

Given an almost disjoint family $\mathcal{A}$, we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. We say $\mathcal{A}$ is *Canjar* if $\mathcal{I}(\mathcal{A})$ is Canjar. In [4] Brendle constructed a non Canjar MAD family under $\mathfrak{b} = \mathfrak{c}$ and asked if it is possible to construct one without additional axioms. We now answer his question in the affirmative.

**Proposition 6.** *There is a non Canjar MAD family.*

*Proof.* Let $\mathcal{P} = \{A_n \mid n \in \omega\}$ be a partition of $\omega$. For every $n \in \omega$ choose $\mathcal{B}_n$ an almost disjoint family of subsets of $A_n$. Construct a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ such that for every $t \in T$ there is $n_t \in \omega$ with the property that $suc(t) = [A_{n_t}]^{<\omega}$ and make sure that if $t \neq s$ then $n_t \neq n_s$, and for every $m$ there is a $t$ such that $n_t = m$. For every branch $b \in [T]$ let $A_b = \bigcup_{n\in\omega} b(n)$ and note that $\mathcal{A} = \{A_b \mid b \in [T]\} \cup \bigcup \{\mathcal{B}_n \mid n \in \omega\}$ is an almost disjoint family and $\mathcal{P} \subseteq \mathcal{I}(\mathcal{A})^{++}$. Let $\mathcal{A}'$ be any MAD family extending $\mathcal{A}$. Note that $\mathcal{P} \subseteq \mathcal{I}(\mathcal{A}')^{+}$ so $T$ is an $\mathcal{I}(\mathcal{A}')^{+}$-tree of finite sets but it has no positive branch. □

Interestingly, we do not know if there is a Canjar MAD family in ZFC. Obviously they exist under $\mathfrak{a} < \mathfrak{d}$. We will now give some sufficient conditions for the existence of a Canjar MAD family. Usually, we will construct a MAD family $\mathcal{A} = \{A_\alpha \mid \alpha \in \kappa\}$ recursively and in such case we will denote by $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$. Call *Part* the set of all interval partitions (partitions in finite sets) of $\omega$. We may define an order on *Part* as follows: given $\mathcal{P}, \mathcal{Q} \in Part$ we say that $\mathcal{P} \leq^* \mathcal{Q}$ if for almost all $Q \in \mathcal{Q}$ there is $P \in \mathcal{P}$ such that $P \subseteq Q$. In [2] it is proved that the smallest size of a dominating family of interval partitions is $\mathfrak{d}$.

First we will give a combinatorial reformulation of $\min\{\mathfrak{d}, \mathfrak{r}\}$.

**Proposition 7.** *If $\kappa$ is an infinite cardinal, then $\kappa < min\{\mathfrak{d}, \mathfrak{r}\}$ if and only if for every $\langle \mathcal{P}_\alpha \mid \alpha \in \kappa\rangle$ family of interval partitions of $\omega$, there is an interval partition $\mathcal{Q} = \{Q_n \mid n \in \omega\}$ with the property that there are disjoint $A, B \in [\omega]^\omega$ such that for all $\alpha < \kappa$, both $\bigcup_{n\in A} Q_n$ and $\bigcup_{n\in B} Q_n$ contain infinitely many intervals of $\mathcal{P}_\alpha$.*

*Proof.* Let $\kappa < \min\{\mathfrak{d}, \mathfrak{r}\}$ and $\langle \mathcal{P}_\alpha \mid \alpha \in \kappa\rangle$ be a family of interval partitions. We may assume that for every $\mathcal{P}_\alpha$ and $n \in \omega$ there is a $\mathcal{P}_\beta$ such that every interval of $\mathcal{P}_\beta$ contains $n$ intervals of $\mathcal{P}_\alpha$. Define $f_\alpha : \omega \longrightarrow \omega$ such that $f_\alpha(n)$ is the left point of $\mathcal{P}_\alpha$ (so $f_\alpha(0) = 0$). Since $\kappa < \mathfrak{d}$, there is $g : \omega \longrightarrow \omega$ such that $g$ is not dominated by any $f_\alpha$, we may as well assume that $g$ is increasing and $g(0) = 0$. Define the interval partition $\mathcal{Q} = \{Q_n \mid n \in \omega\}$ where $Q_n = [g(n), g(n+1))$. Let $M_\alpha$ be the set of all $n \in \omega$ such that $Q_n$ contains an interval of $\mathcal{P}_\alpha$.

**Claim 1.** *$M_\alpha$ is infinite for every $\alpha < \kappa$.*

By the assumption on our family, it is enough to show that each $M_\alpha$ is not empty. Since $g \not\leq^* f_\alpha$, there is $n \in \omega$ such that $f_\alpha(n) < g(n)$. But then it follows that some interval of $\mathcal{P}_\alpha$ must be contained in one $Q_m$ with $m < n$.

Since $\kappa < \mathfrak{r}$, we know that $\{M_\alpha \mid \alpha < \kappa\}$ is not a reaping family, so there are disjoint $A, B \in [\omega]^\omega$ such that $\omega = A \cup B$ and for every $\alpha$, both $M_\alpha \cap A$ and $M_\beta \cap B$ are infinite. It is clear that $A$ and $B$ are the sets we were looking for.

Now we must show that the conclusion of the proposition fails for $\kappa = \mathfrak{d}$ and $\kappa = \mathfrak{r}$. Let $\mathcal{R} = \{M_\alpha \mid \alpha \in \mathfrak{r}\}$ be a reaping family. Define $\mathcal{P}_\alpha$ such that every interval of $\mathcal{P}_\alpha$ contains one point of $M_\alpha$. Assume there is an interval partition $\mathcal{Q} = \{Q_n \mid n \in \omega\}$ and $A, B \in [\omega]^\omega$ as in the proposition. Let $X = \bigcup_{n\in A} Q_n$. Then no $M_\alpha$ reaps $X$, which is a contradiction since $\mathcal{R}$ was a reaping family.

Finally, let $\langle \mathcal{P}_\alpha \mid \alpha \in \mathfrak{d}\rangle$ be a dominating family of partitions and let $\mathcal{Q}$ be any other partition. Then there is a $P_\alpha$ such that every interval of $P_\alpha$ contains two intervals of $\mathcal{Q}$, so obviously there can not be any $A$ and $B$ as required. □

Using the proposition, we may prove the following result.

**Proposition 8.** *If $\mathfrak{d} = \mathfrak{r} = \mathfrak{c}$ then there is a Canjar MAD family of size continuum (In particular, there is one if $\mathfrak{b} = \mathfrak{c}$ or $cov(\mathcal{M}) = \mathfrak{c}$).*

*Proof.* Let $\mathcal{B}$ be a MAD family of size $\mathfrak{c}$. Enumerate $\left\langle \overline{X}_\alpha \mid \omega \leq \alpha < \mathfrak{c} \right\rangle$ the set of decreasing sequences of chains of finite subsets of $\omega$ and let $[\omega]^\omega = \{Y_\alpha \mid \omega \leq \alpha < \mathfrak{c}\}$. We will recursively construct a MAD family $\mathcal{A} = \{A_\alpha \mid \alpha \in \mathfrak{c}\}$ and $\mathcal{P} = \{P_\alpha \mid \alpha \in \mathfrak{c}\}$ such that,

(1) for every $A_\xi \in \mathcal{A}_\alpha$ there is $B_\xi \in \mathcal{B}$ such that $A_\xi \subseteq B_\xi$. In this way, $\mathcal{A}_\alpha$ is almost disjoint but it is not MAD,
(2) if $\overline{X}_\alpha$ is a decreasing sequence of positive sets of $(\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$ then $P_\alpha$ is a pseudointersection,
(3) if $\beta \leq \alpha$ then $P_\alpha \in (\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$,
(4) if $Y_\alpha$ is almost disjoint with $\mathcal{A}_\alpha$ then $A_\alpha \subseteq Y_\alpha$.

It should be obvious that if we manage to do the construction, then we would have built a Canjar MAD family. We start by taking any partition $\{A_n \mid n \in \omega\}$ of $\omega$ in infinite sets. Assume that we have already defined $\mathcal{A}_\alpha$, we will see how to find $A_\alpha$. If $\overline{X}_\alpha$ is not a sequence of elements in $(\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$ then we define $P_\alpha = fin$. Otherwise, (since $\mathfrak{d} = \mathfrak{c}$) we may find $P_\alpha$ a positive pseudointersection.

Now assume that $Y_\alpha$ is almost disjoint with $\mathcal{A}_\alpha$ (if not, take as $Y_\alpha$ any other set almost disjoint from $\mathcal{A}_\alpha$, note there is always one since $\mathcal{A}_\alpha$ is not MAD). Call $\mathcal{D}$ the set of all finite unions of elements of $\mathcal{A}_\alpha$ and for every $\xi \leq \alpha$ and $B \in \mathcal{D}$ define an interval partition $\mathcal{P}_{\xi B} = \{P_{\xi B}(n) \mid n \in \omega\}$ with the following properties:

(1) for every $n \in \omega$ there is $s \subseteq P_{\xi B}(n)$ such that $s \in P_\xi$ and $s \cap B = \emptyset$,
(2) every $P_{\xi B}(n)$ contains an element of $Y_\alpha$.

Since $\langle \mathcal{P}_{\xi B} \mid \xi \leq \alpha \wedge B \in \mathcal{B} \rangle$ has size less than $\max\{\mathfrak{d}, \mathfrak{r}\}$, by the previous result, there is an interval partition $\mathcal{Q} = \{Q_n \mid n \in \omega\}$ and $C, D$ disjoint such that both $\bigcup\limits_{n \in C} Q_n$ and $\bigcup\limits_{n \in D} Q_n$ contains infinitely many intervals of each $\mathcal{P}_{\xi B}$. Define $A'_\alpha = \bigcup\limits_{n \in C} (Q_n \cap Y_\alpha)$, then $A'_\alpha$ satisfies all the requirements except that it may not be contained in some element of $\mathcal{B}$. However, since $\mathcal{B}$ is MAD we may find $B_\alpha \in \mathcal{B}$ such that $A'_\alpha \cap B_\alpha$ is infinite and then we just define $A_\alpha = A'_\alpha \cap B_\alpha$. □

Given an almost disjoint family $\mathcal{A}$, we will denote by $(\mathcal{I}(\mathcal{A})^{<\omega})^{++}$ the set of all $X \in (\mathcal{I}(\mathcal{A})^{<\omega})^+$ such that there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ with the property that each $A_n$ contains infinitely many elements of $X$. Note that if $\mathcal{A}'$ is an almost disjoint family with $\mathcal{A} \subseteq \mathcal{A}'$ and $X \in (\mathcal{I}(\mathcal{A})^{<\omega})^{++}$ then $X \in (\mathcal{I}(\mathcal{A}')^{<\omega})^+$. The purpose of this definition is the following: assume that we want to construct (recursively) $\mathcal{A} = \{A_\alpha \mid \alpha \in \kappa\}$ a Canjar MAD family, at some stage $\alpha$ of the construction, we may look at some decreasing sequence $\langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$ and somehow we manage to find $P_\alpha$ a pseudointersection with $P_\alpha \in (\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$, we must make sure that $P$ remains positive in the future extensions of $\mathcal{A}_\alpha$. In the previous proof, we made sure that at each step of the construction, we preserved the positiveness of all the $P_\alpha$. Another approach would be to make sure that $P_\alpha \in (\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^{++}$.

**Lemma 3.** *If $\mathcal{A}$ is an almost disjoint family such that for every decreasing sequence $\langle X_n \mid n \in \omega \rangle$ of $(\mathcal{I}(\mathcal{A})^{<\omega})^{+}$ there is a pseudointersection $P \in (\mathcal{I}(\mathcal{A})^{<\omega})^{++}$, then $\mathcal{A}$ is a Canjar MAD family.*

*Proof.* The proof is left to the reader. □

**Lemma 4.** *Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ be an almost disjoint family and let $\langle X_n \mid n \in \omega \rangle$ in $(\mathcal{I}(\mathcal{A})^{<\omega})^{+}$ be a decreasing sequence. Then there is an increasing $f : \omega \longrightarrow \omega$ such that for every $n \in \omega$ there is $s_n \in \wp(f(n) - f(n-1)) \cap X_n$ and $s_n \cap (A_0 \cup \ldots \cup A_n) = \emptyset$ (for ease of writing, assume that $f(-1) = 0$).*

*Proof.* Easy. □

Moreover, note that $f$ can be obtained in a completely definable way. We must also remark that if we define $P = \bigcup\limits_{n\in\omega} X_n \cap \wp(f(n))$ and $B = \bigcup\limits_{n\in\omega} (f(n) - A_0 \cup \ldots \cup A_n)$ then $P$ will be a positive pseudointersection of $\{X_n : n \in \omega\}$, $B$ will contain infinitely many elements of $P$ and $\mathcal{A} \cup \{B\}$ will be an AD family.

The following guessing principle was defined in [17],

- **$\diamondsuit(\mathfrak{b})$:** For every Borel coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ there is a $G : \omega_1 \longrightarrow \omega^\omega$ such that for every $R \in 2^{\omega_1}$ the set $\{\alpha \mid C(R \restriction \alpha) \ {}^*\!\not\geq G(\alpha)\}$ is stationary (such $G$ is called a *guessing sequence for $C$*).

Recall that a coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ is *Borel* if for every $\alpha$, the function $C \restriction 2^\alpha$ is Borel. It is easy to see that $\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{b} = \omega_1$ and in [17] it is proved that it also implies $\mathfrak{a} = \omega_1$.

**Proposition 9.** *Assuming $\diamondsuit(\mathfrak{b})$, there is a Canjar MAD family.*

*Proof.* For every $\alpha < \omega_1$ fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. With a suitable coding, the coloring $C$ will be defined on pairs $t = (\mathcal{A}_t, X_t)$ where $\mathcal{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t = \langle X_n \mid n \in \omega \rangle$. We define $C(t)$ to be the constant 0 function in case $\mathcal{A}_t$ is not an almost disjoint family or if $X_t$ is not a decreasing sequence of $(\mathcal{I}(\mathcal{A})^{<\omega})^{+}$. In the other case, let $C(t)$ be the function obtained by the previous lemma with $\mathcal{A} = \{A_{\alpha_n} \mid n \in \omega\}$ and $X_t$. Using $\diamondsuit(\mathfrak{b})$, let $G : \omega_1 \longrightarrow \omega^\omega$ be a guessing sequence for $C$. By changing $G$ if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^* G(\beta)$.

We will now define our MAD family: start by taking $\{A_n \mid n \in \omega\}$ a partition of $\omega$. Having defined $A_\xi$ for all $\xi < \alpha$, we proceed to define

$$A_\alpha = \bigcup_{n\in\omega} (G(\alpha)(n) - A_{\alpha_0} \cup \ldots \cup A_{\alpha_n})$$

in case this is an infinite set, otherwise take any $A_\alpha$ that is almost disjoint from $\mathcal{A}_\alpha$. We will see that $\mathcal{A}$ is a Canjar MAD family. Let $X = \langle X_n \mid n \in \omega \rangle$ be a decreasing sequence in $(\mathcal{I}(\mathcal{A})^{<\omega})^{+}$. Consider the branch $R = (\langle A_\xi \mid \xi < \omega_1 \rangle, X)$ and pick $\beta^0, \beta^1, \beta^2, \ldots$ such that $C(R \restriction \beta^n) \ {}^*\!\not\geq G(\beta^n)$. Choose $\alpha$ bigger than all the $\beta^n$ and define $h = G(\alpha)$ and $P = \bigcup_{n\in\omega} \wp(h(n)) \cap X_n$. It is clear that $P$ is a pseudointersection of $X$. We will now just show that $P \in (\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^{++}$ and we will do this by proving that each $A_{\beta^n}$ contains infinitely many elements of $P$.

Fix $n \in \omega$ and Let $t = R \restriction \beta^n$. Since $C(t) \ {}^*\!\not\geq G(\beta^n)$ we may find $m$ such that $C(t)(m) < G(\beta^n)(m) < h(m)$. In such case (by the property of $C(t)$) there is $s \in \wp(C(t)(m)) \cap X_m$ disjoint from $A_{\beta^n_0}, \dots A_{\beta^n_m}$ and then $s \subseteq A_{\beta^n}$ and $s \in P$. $\square$

We quote an instance of a very general theorem from [17].

**Proposition 10** ([17])**.** *Let $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\wp(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathbb{P}_{\omega_2} \Vdash$ "$\mathfrak{b} = \omega_1$" then $\mathbb{P}_{\omega_2} \Vdash$ "$\diamondsuit(\mathfrak{b})$".*

With the aid of the previous result, we can prove that there are Canjar MAD families in many of the models obtained by countable support iteration.

**Corollary 2.** *Let $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\wp(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. Let $G \subseteq \mathbb{P}_{\omega_2}$ be generic, then there is a Canjar MAD family in $V[G]$.*

*Proof.* If in $V[G]$ happens that $\mathfrak{b}$ is $\omega_2$ then we already know there is a Canjar MAD family. Otherwise $\mathfrak{b} = \omega_1$ and then $\diamondsuit(\mathfrak{b})$ holds in $V[G]$ so there is a Canjar MAD family. $\square$

Recall that a forcing is $\omega^\omega$*-bounding* if it does not add unbounded reals (or, equivalently, the ground model reals still form a dominating family). Given a forcing $\mathbb{P}$ and a Canjar MAD family $\mathcal{A}$, we say that $\mathcal{A}$ is $\mathbb{P}$ MAD-*Canjar indestructible* if it remains Canjar MAD after forcing with $\mathbb{P}$. We will see that under $\mathsf{CH}$, no proper $\omega^\omega$-bounding forcing of size $\omega_1$ can destroy all Canjar MAD families. If $\mathbb{P}$ is a partial order, $\dot{a}$ is a $\mathbb{P}$ name and $G \subseteq \mathbb{P}$ is a generic filter, we will denote by $\dot{a}[G]$ the evaluation of $\dot{a}$ according to the generic filter $G$.

**Proposition 11.** *Assume $\mathsf{CH}$ and let $\mathbb{P}$ be a proper $\omega^\omega$-bounding forcing of size $\omega_1$. Then there is a $\mathbb{P}$ MAD-Canjar indestructible family.*

*Proof.* Using the Continuum Hypothesis and the properness of $\mathbb{P}$, we may find a set $H = \{(p_\alpha, \dot{W}_\alpha) \mid \alpha \in \omega_1\}$ such that for all $p$ and $\dot{X}$, if $p$ forces that $\dot{X}$ is a decreasing sequence, then there is $\alpha$ such that $p \leq p_\alpha$ and $p_\alpha \Vdash$ "$\dot{W}_\alpha = \dot{X}$".

We will construct a MAD family $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ such that if $p_\alpha$ forces that $\dot{W}_\alpha$ is a decreasing sequence of positive sets in $(\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$, then there is $q \leq p_\alpha$ with the property that there is $\dot{P}_\alpha$ such that $q$ forces that $\dot{P}_\alpha$ is a pseudointersection of $\dot{W}_\alpha$ and that $\dot{P}_\alpha$ is in $(\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^{++}$ (hence $q$ will force that $\dot{P}_\alpha$ is in $(\mathcal{I}(\mathcal{A})^{<\omega})^+$).

First take $\{A_n \mid n \in \omega\}$ a partition of $\omega$. Assume that we have defined $\mathcal{A}_\alpha$. We will see how to define $\mathcal{A}_{\alpha+\omega}$. In case $p_\alpha$ does not force that $\dot{W}_\alpha$ is a decreasing sequence of positive sets in $(\mathcal{I}(\mathcal{A}_\alpha)^{<\omega})^+$ then take $\mathcal{A}_{\alpha+\omega}$ be any almost disjoint family extending $\mathcal{A}_\alpha$. Now assume otherwise, write $\alpha = \{\alpha_n \mid n \in \omega\}$ and let $G \subseteq \mathbb{P}$ be a generic filter with $p_\alpha \in G$. Since $\mathcal{A}_\alpha$ is countable and $\dot{W}_\alpha[G] = \left\langle \dot{W}_\alpha(n)[G] \mid n \in \omega \right\rangle \in V[G]$ is a sequence of positive sets in $V[G]$, there is an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\} \in V[G]$ such that for all $n \in \omega$, there is $s_n \subseteq P_n$ such that $s_n \in \dot{W}_\alpha(n)[G]$ and $s_n$ is disjoint from $A_{\alpha_0} \cup \ldots \cup A_{\alpha_n}$. Define $P_\alpha = \bigcup \left(P_n \cap \dot{W}_\alpha(n)[G]\right)$. Let $q' \leq p_\alpha$ force that $\dot{\mathcal{P}}$ is an interval partition and every $\dot{P}_n$ contains an element in $\dot{W}_\alpha(n)$ disjoint from $A_{\alpha_0} \cup \ldots \cup A_{\alpha_n}$. Since $\mathbb{P}$ is $\omega^\omega$-bounding, there is $q \leq q'$ and $\mathcal{Q} = \{Q_n \mid b \in \omega\}$ a ground model partition such that $q \Vdash$ "$\dot{\mathcal{P}} \leq \mathcal{Q}$". Let $\{D_n \mid n \in \omega\}$ be a partition of $\omega$ with $D_n = \left\{d_n^i \mid i \in \omega\right\}$. Define $A_{\alpha+n} = \bigcup_{n\in\omega} \left(P_{d_n^i} - A_{\alpha_0} \cup \ldots A_{\alpha_n}\right)$, then $\mathcal{A}_{\alpha+\omega}$ is an AD family and $q$ forces that each $A_{\alpha+n}$ contains infinitely many elements of $\dot{P}_\alpha$. $\square$

**Corollary 3.** *There are Canjar MAD families in the Cohen, Random, Hechler, Sacks, Laver, Miller and Mathias model.*

*Proof.* We have already proved it for the models obtained by countable support iteration and in the Cohen and Hechler models since $\mathrm{cov}(\mathcal{M})$ is equal to $\mathfrak{c}$. It only remains to check it for the Random real model. Assume $\mathsf{CH}$ and denote by $\mathbb{B}(\kappa)$ the forcing notion for adding $\kappa$ random reals. Let $G \subseteq \mathbb{B}(\omega_2)$ be a generic filter, we want to see that there is a Canjar MAD family in $V[G]$. By the previous proposition, we know there is $\mathcal{A}$ a $\mathbb{B}(\omega_1)$ MAD-Canjar indestructible family. It is easy to see that $\mathcal{A}$ is $\mathbb{B}(\omega_2)$ MAD-Canjar indestructible (since every new real in $V[G]$ appears in an intermediate extension after adding only $\omega_1$ random reals). $\square$

Although there still may be models without Canjar MAD families, it is easy to show that there are always uncountable Canjar almost disjoint families. Let $C_n = \{n\} \times \omega$ and given a family of increasing functions $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\} \subseteq \omega^\omega$ such that if $\alpha < \beta$ then $f_\alpha <^* f_\beta$ define $\mathcal{A}_\mathcal{B} = \mathcal{B} \cup \{C_n \mid n \in \omega\}$ and note that it is an almost disjoint family.

**Proposition 12.** *There is a family $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ such that $\mathcal{A}_\mathcal{B}$ is Canjar, so there is an uncountable Canjar almost disjoint family.*

*Proof.* If $\omega_1 < \mathfrak{d}$ then any $\mathcal{B}$ will do, so assume that $\mathfrak{d} = \omega_1$. Let $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ be a well-ordered dominating family. For every $\alpha < \omega_1$ define $L_\alpha = \{(n,m) \mid m < f_\alpha(n)\}$ and for a given $X$ define $X(\alpha) = X \cap [L_\alpha]^{<\omega}$. We will show that $\mathcal{I}(\mathcal{A}_\mathcal{B})^{<\omega}$ is a $P^+$-ideal and to show that, we will need the following "reflection property" due to Nyikos (see [18]),

**Claim 2.** *If $X \in (\mathcal{I}(\mathcal{A})^{<\omega})^+$ then $X(\alpha) \in (\mathcal{I}(\mathcal{A})^{<\omega})^+$ for some $\alpha < \omega_1$.*

Assume this is not the case, so for every $\alpha < \omega_1$ the set $X(\alpha) \in \mathcal{I}(\mathcal{A}_\mathcal{B})^{<\omega}$, which means there is $F_\alpha \in [\alpha]^{<\omega}$ and $n_\alpha \in \omega$ such that $Z_\alpha = \bigcup_{\xi \in F_\alpha} f_\xi \cup \bigcup_{i \leq n_\alpha} C_i$ intersects every element of $X(\alpha)$. By a trivial application of elementary submodels, there are $S \subseteq \omega_1$ a stationary set, $F$ a finite subset of $\omega_1$ and $n \in \omega$ such that $F = F_\alpha$ and $n_\alpha = n$ for every $\alpha \in S$, call $Z = \bigcup_{\xi \in F} f_\xi \cup \bigcup_{i \leq n} C_i \in \mathcal{I}(\mathcal{A}_\mathcal{B})$.

Given $s \subseteq \omega \times \omega$, define $\pi(s) = \{n \mid \exists m ((n,m) \in s)\}$. As $X \in (\mathcal{I}(\mathcal{A}_\mathcal{B})^{<\omega})^+$ we may find a sequence $Y = \{x_n \mid n \in \omega\} \subseteq X$ such that $x_n \cap Z = \emptyset$ and $\max(\pi(x_n)) < \min(\pi(x_{n+1}))$ for all $n \in \omega$. Since $\mathcal{B}$ is a well-ordered dominating family of increasing functions, there is $\alpha \in S$ such that the set $Y \cap L_\alpha$ is infinite. Note that $Z_\alpha = Z$ so $x_n \cap Z_\alpha = \emptyset$ for all $x_n \in Y \cap L_\alpha$ which contradicts the choice of $F_\alpha$ and $n_\alpha$.

We are ready to show that $\mathcal{I}(\mathcal{A}_\mathcal{B})^{<\omega}$ is a $P^+$-ideal. Let $\langle X_n \mid n \in \omega \rangle$ be a decreasing sequence of positive sets. Find $\alpha$ such that $X_n(\alpha) \in (\mathcal{I}(\mathcal{A})^{<\omega})^+$ for all $n \in \omega$ (this is possible because if $\beta < \gamma$ and $X_n(\beta)$ is positive $X_n(\gamma)$ is positive). Let $\alpha = \{\alpha_n \mid n \in \omega\}$. For every $n \in \omega$ choose $x_n \in X_n$ such that $x_n$ is disjoint from $\bigcup_{i \leq n} f_{\alpha_i} \cup \bigcup_{i \leq n} C_i$ then it is easy to see that $X = \{x_n \mid n \in \omega\}$ is a positive pseudointersection. □

In particular,

**Corollary 4.** *There is a non Borel Canjar ideal generated by $\omega_1$ sets.*

*Proof.* By the previous result, we know there is $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\}$ such that $\mathcal{I}(\mathcal{A}_\mathcal{B})$ is Canjar, it is enough to show it is not $F_\sigma$. Assume otherwise, so it must be $F_\sigma$. Let $\mathcal{I}(\mathcal{A}_\mathcal{B}) = \bigcup_{n \in \omega} C_n$ where each $C_n$ is a compact set. Clearly, there is $n \in \omega$ such that $C_n$ contains uncountably many elements of $\mathcal{B}$. Note that $C_n \cap \mathcal{B} = C_n \cap \omega^\omega$ so $A = C_n \cap \mathcal{B}$ is a Borel set. For a given $Z$ subset of a Polish space, recall the following definition (see [21])

> $\mathsf{OCA}(Z)$**:** If $c : Z^2 \longrightarrow 2$ is a coloring such that $c^{-1}(0)$ is open, then either $Z$ has an uncountable 0-monochromatic set, or $Z$ is the union of countable many 1-monochromatic sets.

In [21] it is proved that $\mathsf{OCA}(Z)$ is true for every analytic set, so in particular $\mathsf{OCA}(A)$ is true. However, we will arrive to a contradiction using the same argument that $\mathsf{OCA}$ implies that $\mathfrak{b} = \omega_2$ (see [21]). □

## 5. Ideals generated by branches

If $b \in 2^\omega$ we denote by $\widehat{b} = \{b \restriction n \mid n \in \omega\}$. Let $A$ be a dense, co-dense subset of $2^\omega$. We define $\mathcal{I}_A$ the *branching ideal of* $A$ as the set of all $X \subseteq 2^{<\omega}$ such that there are $b_1, \ldots, b_n \in A$ with the property that $X \subseteq \widehat{b_1} \cup \ldots \cup \widehat{b_n}$. Clearly, if $M \in [\widehat{b}]^\omega$ with $b \notin A$ then $M \in \mathcal{I}_A^+$, and also every infinite antichain, is positive.

**Lemma 5.** *$\mathcal{I}_A$ is $P^+$ for every $A \subseteq 2^\omega$.*

*Proof.* This result follows since $\mathcal{I}_A$ is the ideal generated by an infinite almost disjoint family. □

We will now investigate when $\mathcal{I}_A$ is $P^+(tree)$ and Canjar.

**Proposition 13.** *If $A$ is the union of less than $\mathfrak{d}$ compact sets, then $\mathcal{I}_A$ is Canjar.*

*Proof.* Assume that $A = \bigcup_{\alpha<\kappa} C_\alpha$ where $C_\alpha$ is compact and $\kappa < \mathfrak{d}$ moreover, we may assume that for every $b_1, \ldots, b_n \in A$ there is a $C_\alpha$ such that $b_1, \ldots, b_n \in C_\alpha$. We will show that $\mathcal{I}_A^{<\omega}$ is a $P^+$-ideal. Before starting the proof we must do an important observation: assume that $Y \in (\mathcal{I}_A^{<\omega})^+$ and for every $a \in Y$ define $U_a = \{b \in 2^\omega \mid a \cap \widehat{b} = \emptyset\}$ and since $a$ is finite then $U_a$ is open and $\langle U_a \mid a \in Y\rangle$ is an open cover of $A$. Therefore, every $C_\alpha$ is contained in only a finite number of $U_a$.

Let $\langle X_n \mid n \in \omega\rangle$ be a decreasing family of positive sets of $\mathcal{I}_A^{<\omega}$. For every $\alpha < \kappa$ we define $f_\alpha : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that for every if $n \in \omega$ then $f_\alpha(n) \subseteq X_n$ and $C_\alpha \subseteq \bigcup_{a \in f_\alpha(n)} U_a$. Since $\kappa < \mathfrak{d}$, there is $f : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that $f(n) \subseteq X_n$ and for all $\alpha < \kappa$ it happens that $f_\alpha(n) \subseteq f(n)$ for infinitely many $n \in \omega$. It is easy to see that $\bigcup_{n\in\omega} f(n)$ is a positive pseudointersection of $\langle X_n \mid n \in \omega\rangle$. □

Given a topological space $X$, we say that an open cover $\mathcal{U}$ is an *$\omega$-cover* if for every $x_0, \ldots, x_n \in X$ there is $U \in \mathcal{U}$ such that $x_0, \ldots, x_n \in \mathcal{U}$. We say that $X$ is $S_{fin}(\Omega, \Omega)$ if for every sequence $\langle \mathcal{U}_n \mid n \in \omega\rangle$ of $\omega$-covers, there are $F_n \in [\mathcal{U}_n]^{<\omega}$ such that $\bigcup_{n\in\omega} F_n$ is an $\omega$-cover (see [19] for more information concerning this type of spaces). The following was noted by Ariet Ramos.

**Proposition 14.** *$\mathcal{I}_A$ is Canjar if and only if $A$ is $S_{fin}(\Omega, \Omega)$.*

*Proof.* First assume that $A$ is $S_{fin}(\Omega, \Omega)$ and let $\langle X_n \mid n \in \omega\rangle \subseteq (\mathcal{I}_A^{<\omega})^+$ be a decreasing sequence. Given any $a$ we define $U_a = \{b \mid a \cap \widehat{b} = \emptyset\}$. Since each $X_n$ is positive, $\mathcal{V}_n = \{U_a \mid a \in X_n\}$ is an $\omega$-cover of $A$. In this way, $\langle \mathcal{V}_n \mid n \in \omega\rangle$ is a sequence of $\omega$-covers, so there are $F_n \in [X_n]^{<\omega}$ such that $\{U_a \mid a \in \bigcup_{n\in\omega} F_n\}$ is an $\omega$-cover. It is easy to see that $P = \bigcup_{n\in\omega} F_n$ is a positive pseudointersection of $\langle X_n \mid n \in \omega\rangle$.

Now, assume that $\mathcal{I}_A$ is Canjar and let $\langle \mathcal{U}_n \mid n \in \omega\rangle$ be a sequence of $\omega$-covers. Given an open set $U$, define $Y_U = \{a \mid \forall b(\widehat{b} \cap a = \emptyset \longrightarrow b \in U)\}$. Define $X_n = \bigcup_{U\in\mathcal{U}_n} Y_U$. Since $\mathcal{U}_n$ is an $\omega$-cover, each $X_n$ is positive. Since $\mathcal{I}_A$ is Canjar, there are $F_n \in [X_n]^{<\omega}$ such that $P = \bigcup_{n\in\omega} F_n$ is a positive pseudointersection. For every $a \in F_n$ choose $U_a \in \mathcal{U}_n$ with the property that $a \in Y_{U_a}$. It is not difficult to check that $\{U_a \mid a \in F_n \wedge n \in \omega\}$ is an $\omega$-cover. □

Given an ideal $\mathcal{I}$ we define $\mathcal{LF}(\mathcal{I})$ the *Laflamme Game on $\mathcal{I}$* as follows,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I | $X_0$ | | $X_1$ | | $X_2$ | | $X_3$ | $\cdots$ |
| II | | $s_0$ | | $s_1$ | | $s_2$ | | $\cdots$ |

where each $X_n \in \mathcal{I}^+$ and $s_n$ is a finite subset of $X_n$. The player $II$ wins the game if $\bigcup s_n \in \mathcal{I}^+$. Laflamme proved in [13] that $\mathcal{I}$ is a $P^+$ (*tree*) ideal if and only if player $I$ does not have a winning strategy in $\mathcal{LF}(\mathcal{I})$. In case of branching ideals, the Laflamme game can be simplified. Given $A \subseteq 2^\omega$ define the game $\mathcal{LF}'(\mathcal{I})$ as follows,

| I | $b_0$ | | $b_1$ | | $b_2$ | | $b_3$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| II | | $s_0$ | | $s_1$ | | $s_2$ | | $\cdots$ |

where each $b_n \notin A$, $s_n$ is an initial segment of $b_n$, $s_n \subsetneq s_{n+1}$ and $b_{n+1} \in \langle s_n \rangle$. The player $II$ wins the game if $\bigcup s_n \notin A$. The analogue of the result of Laflamme is the following.

**Proposition 15.** *$\mathcal{I}_A$ is a $P^+(tree)$ ideal if and only if player $I$ does not have a winning strategy in $\mathcal{LF}'(\mathcal{I})$.*

*Proof.* It is easy to see that if $I$ has a winning strategy in $\mathcal{LF}'(\mathcal{I})$ then she has one in $\mathcal{LF}(\mathcal{I})$ so $\mathcal{I}$ is not $P^+(tree)$. For the other direction, assume that $I$ does not have a winning strategy and let $T$ be a $\mathcal{I}_A^+$ tree. We will show that there is $b \in [T]$ such that $\bigcup b \restriction n \in \mathcal{I}_A^+$.

**Case 1.** For all $s \in T$ and $n \in \omega$ there is $t$ an extension of $s$ such that $\bigcup\limits_{i<|t|} t \restriction i$ can not be covered by $n$ branches.

In this case, we simply choose $s_0, s_1, \dots$ such that $s_{n+1}$ extends $s_n$ and it can not be covered by $n$ branches. It is clear that $b = \bigcup s_n$ is as desired.

**Case 2.** Without loss of generality, there is $n \in \omega$ such that for every $t \in T$, the set $\bigcup\limits_{i<|t|} t \restriction i$ can be covered by $n$ branches.

By an easy compactness argument, for every $s \in T$ there are $b_0^s, \dots, b_{n-1}^s \in 2^\omega$ such that $X_s \subseteq \widehat{b}_0^s \cup \ldots \cup \widehat{b}_{n-1}^s$, $b_0^s \notin A$ and $X_s \cap \widehat{b}_0^s$ is infinite. Let $T' \subseteq T$ such that for every $t \in T'$ there is $m_t$ with the property that $t = X_t \cap 2^{\leq m_t}$.

We say that $s$ *prefers* $t$ if $s$ extends $t$, $m_s > m_t$ and $b_0^s \in \langle b_0^t \restriction m_t \rangle$. We also say that $t$ is *totally preferred* if for all $s \leq t$ there is $s' \leq s$ such that $s'$ prefers $t$. We first claim that there is $t \in T$ that is totally preferred. Assume this is not the case, then we do the following:

(1) Let $t_\emptyset = \emptyset$.
(2) Let $t_1 \leq t_0$ such that no extension of $t_1$ prefers $t_0$.
(3) Let $t_2 \leq t_1$ such that no extension of $t_2$ prefers $t_1$.
(4) $\vdots$

We keep this procedure until we find $t_{n+1}$, but then $t_{n+1}$ must prefer some $t_i$ (with $i \leq n$) which is a contradiction. Now assume $t$ is totally preferred, we will describe $\pi$ an strategy for player $I$.

(1) First, player $I$ plays $b_0^t$,

(2) if player $II$ plays $s_0$, then $I$ finds $n_0 \geq |s_0|, \Delta(X_t)$ and let $t_0 = X_t \cap 2^{\leq n_0}$. Player $I$ finds $t_0' \leq t_0$ such that $t_0'$ prefers $t$ and $I$ plays $b_0^{t_0'}$.
(3) if player $II$ plays $s_1$, then $I$ finds $n_1 \geq |s_1|, \Delta\left(X_{t_0'}\right)$ and let $t_1 = X_{t_0'} \cap 2^{\leq n_1}$. Player $I$ finds $t_1' \leq t_1$ such that $t_1'$ prefers $t$ and $I$ plays $b_0^{t_1'}$.
(4) $\vdots$

since $\pi$ is not a winning strategy, there are $s_0, s_1, s_2, \ldots$ such that if player $II$ play $s_n$ at round $n$ then he will win in case $I$ follows $\pi$. Let $d = (\pi(s_0, \ldots, s_i) \upharpoonright n_i)$. Then $\bigcup d \notin A$ (since $II$ won the game) and $d$ is a branch through $T$. $\square$

We will now give a topological characterization of the sets such that its branching ideal is $P^+(tree)$. Recall that a topological space is a *Baire space* if no non-empty open sets are meager, and a space is called *completely Baire* if all of its closed subsets are Baire. Hurewicz proved that a space is completely Baire if and only if it does not contain a closed copy of $\mathbb{Q}$ (see [22] pages 78 and 79).

**Proposition 16.** *$\mathcal{I}_A$ is $P^+(tree)$ if and only if $2^\omega - A$ is completely Baire.*

*Proof.* Assume that $\mathcal{I}_A$ is $P^+(tree)$ and suppose that $2^\omega - A$ is not completely Baire, so there is a perfect set $C$ such that $A \cap C = \{d_n \mid n \in \omega\}$ is countable dense in $C$. Consider the following strategy $\pi$ for $I$ in $\mathcal{LF}'(2^\omega - A)$.

(1) $I$ plays $d_0$,
(2) if $II$ plays $s_0$, then $I$ plays $d_{n_1}$ where $n_1 = \min\{i > 0 \mid d_i \in \langle s_0 \rangle\}$,
(3) if $II$ plays $s_1$, then $I$ plays $d_{n_2}$ where $n_1 = \min\{i > n_1 \mid d_i \in \langle s_1 \rangle\}$,
(4) $\vdots$

Since this is not a winning strategy, there are $s_0, s_1, s_2, \ldots$ such that if $I$ follows $\pi$ and $II$ plays $s_i$ at the round $i$, then $II$ will win. Let $a = \bigcup\limits_{n \in \omega} s_n$. Then $a \in A \cap C$ since $C$ is compact and $II$ won the game, however, $a$ is different than all the $d_n$, which is a contradiction.

Now assume that $A \cap C$ is uncountable whenever $C$ is perfect and $A \cap C$ is dense in $C$. Aiming for a contradiction, assume that $I$ has $\pi$ a winning strategy in $\mathcal{LF}'(2^\omega - A)$. Let $D \subseteq 2^\omega$ be the set of all $b \in 2^\omega$ such that there are $s_0, s_1, \ldots, s_n$ with the property that $\pi(s_0, s_1, \ldots, s_n) = b$. Since $\pi$ is a winning strategy, $D \subseteq A$ has no isolated points and $C = \overline{D}$ is perfect. Since $D$ is countable, there is $b \in A \cap C - D$. Note that $b$ corresponds to a legal play in $\mathcal{LF}'(2^\omega - A)$ in which $II$ won (since $b \in A$) which is a contradiction. $\square$

For our next result, we need to recall a result from Kechris, Louveau and Woodin ([10], see also [11] Theorem 21.22).

**Proposition 17** ([10])**.** *If $A \subseteq 2^\omega$ is analytic and $A \cap B = \emptyset$ then one of the following holds,*

(1) *there is $F$ an $F_\sigma$ set such that separates $A$ from $B$ or,*

(2) *there is a perfect set $C \subseteq A \cup B$ such that $C \cap B$ is countable dense in $C$.*

With this we can easily prove the following.

**Corollary 5.** *If $A$ is Borel and is not $F_\sigma$ then $\mathcal{I}_A$ is not $P^+(tree)$.*

*Proof.* If $A$ is Borel but not $F_\sigma$ then, by the Kechris-Louveau-Woodin theorem, there is a perfect set $C$ such that $C \cap (2^\omega - A)$ is countable dense in $C$, which shows that $\mathcal{I}_A$ is not $P^+(tree)$. □

An alternative proof of the previous corollary would be to note that if $A$ is Borel but not $F_\sigma$ then $\mathcal{I}_A$ will also be Borel but not $F_\sigma$, so it can not be $P^+(tree)$. The next result will give us an example of a non Canjar ideal that is $P^+(tree)$,

**Proposition 18.** *If $B$ is Bernstein then $\mathcal{I}_B$ is $P^+(tree)$ but not Canjar.*

*Proof.* Since the complement of a Bernstein set is Bernstein, it follows easily by the topological characterization of $P^+(tree)$ that $\mathcal{I}_B$ is $P^+(tree)$. We will now show it is not Canjar. Build an increasing sequence $\langle \mathcal{C}_n \mid n \in \omega \rangle$ of compact sets in the following way,

(1) we choose $b_0^0 \notin B$ and let $\mathcal{C}_0 = \{\widehat{b_0^0}\}$,
(2) we choose $\langle b_n^{01} \rangle_{n\in\omega} \subseteq 2^\omega - B$ a convergent sequence to $b_0^0$ and define $\mathcal{C}_1 = \mathcal{C}_0 \cup \{\widehat{b_n^{01}} \mid n \in \omega\}$,
(3) for every $b_n^{01}$ we choose $\langle b_n^{012} \rangle_{n\in\omega} \subseteq 2^\omega - B$ a convergent sequence to $b_n^{01}$ and define $\mathcal{C}_2 = \mathcal{C}_1 \cup \{\widehat{b_n^{012}} \mid n \in \omega\}$,
(4) $\vdots$

It is clear that each $\mathcal{C}_n \subseteq \mathcal{I}_B^+$ and $\langle \mathcal{C}_n : n \in \omega \rangle$ forms an increasing sequence of compact sets. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be a finite partition of $2^{<\omega}$ and define $D$ as the set of all $x \in 2^\omega$ such that there is $\langle d_n \mid n \in \omega \rangle$ with the coherence property with respect to $\mathcal{P}$ and $\widehat{x} \cap P_n = \widehat{d_n}$. It is easy to see that $D$ is an uncountable closed set, so $B \cap D \neq \emptyset$ and hence $\mathcal{I}_B$ is not Canjar. □

Recall that a *Luzin set* is an uncountable set that has countable intersection with every meager set. Luzin sets exist under $\mathsf{CH}$ or after adding at least $\omega_1$ Cohen reals. However, it is easy to see that the existence of a Luzin set implies that $\mathrm{non}(\mathcal{M})$ is $\omega_1$, so their existence is not provable from $\mathsf{ZFC}$. By a suitable modification of the previous argument, one can show the following.

**Corollary 6.** *If $L$ is a (dense) Luzin set, then $\mathcal{I}_{\omega-L}$ is not Canjar.*

## 6. Open Questions

There are some questions we were unable to answer, probably the most interesting one is the following.

**Problem 1.** *Is there a Canjar MAD family? Is there one of cardinality continuum?*

We proved that if $\mathfrak{d} = \mathfrak{r} = \mathfrak{c}$ then there is a Canjar MAD family of size continuum, but we do not even know the answer to the following question.

**Problem 2.** *Does $\mathfrak{d} = \mathfrak{c}$ implies there is a Canjar MAD family?*

The characterization of Canjar ideals suggest the next questions.

**Problem 3.** *Are there coherent strong $P^+$-ideals that are not strong $P^+$?*

We know there are $P^+$-ideals that are not $P^+(tree)$, but we do not know the answer of the following question.

**Problem 4.** *Is there a Canjar ideal $\mathcal{I}$ such that $\mathcal{I}^{<\omega}$ is not $P^+(tree)$?*[4]

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